How to Say True Things in Algebraic Topology

Emily Grosholz

Department of Philosophy, 240 Sparks Building

The Pennsylvania State University, University Park, PA 16802 USA

Email: erg2@psu.edu

I. Two Roles for Representations

I count myself as one of the current generation of philosophers of science who criticize semantical philosophy of science and call for the introduction of pragmatical as well as semantical and syntactical concerns into our account of scientific rationality. This means that we must look at how representations and interventions in science and mathematics are used in a particular historical and theoretical context. In mathematics, I would argue, this context is characterized first and foremost by a collection of solved and unsolved problems, a tradition of kinds of problematic items and modes of representation for them, methods (including calculation and construction procedures) for addressing the problems, strategies for moving among distinct domains and between the finitary and infinitary, and standards for what may be a satisfactory solution to a problem.[1]

Summarizing his arguments against the primary epistemic role given to the notion of 'isomorphism' in the semantical account of truth as a relation among a theory and its models, my fellow "pragmatist," the philosopher of chemistry Robin Hendry writes, "Firstly, representation cannot be identified with isomorphism, because there are just too many relation-instances of isomorphism. Secondly, a particular relation-instance of isomorphism is a case of representation only in the context of a scheme of use that fixes what is to be related to what, and how. Thirdly, in reacting to the received [syntactic] view's linguistic orientation, the semantic view goes too far in neglecting language, because language is a crucial part of the context that makes it possible to use mathematics to represent. Natural languages afford us abilities to refer, and equations borrow these abilities. We cannot fully understand particular cases of representation in the absence of a 'natural history' of the traditions of representation of which they are a part.' [2]

When we try to speak truly about things, we ask language (broadly construed) to perform at least two different roles: to indicate what we are talking about, and to analyze it by discovering its conditions of intelligibility, its rational requisites, its reasons for being what it is. [3] In the passage just cited, Hendry argues that the semantical approach fails to account for how language fulfills both these functions in successful science and mathematics. The traditional schema for a proposition used by logicians is ‘S is P,’ and in general the subject term pertains to the first role, and the predicate term to the second. In two contexts drawn from an important late twentieth century textbook on algebraic topology that I examine in this essay, we find the first role played both by icons and by standard notations that have become proper names of certain objects; importantly, these designations are always accompanied by natural language that explain them and relate them to more abstract levels in the discourse. Mathematicians don’t ask icons to do the work of designating mathematical things all by themselves; an icon is not supposed to be “self-evident.” [4]

We find the second role played by a repertoire of representations that, to be most effective, act in tandem to carry out an analysis. Here I borrow the term from Leibniz: it may be variously thought of as the search for the conditions of solvability of problems, and as the search for conditions of intelligibility of the things involved in the problems—what requisites make them necessarily what they are. [5] In the case of De Rham’s Theorem, where manifolds are “triangulated,” we find a single representation called upon to be both a finite-combinatorial item and an infinitary-continuous item depending upon how it is read. Natural language then also helps to explain the relation between the different modes of representation, and the controlled ambiguity of a single mode when the relation between its disparate uses must be clarified. The important asymmetry between S and P in the assertion of a proposition has been covered over by 20th century logic. Language used for S typically plays the role of referring; language used for P typically plays the role of analyzing; the assertion of a proposition juxtaposes them.

Thus, there is a deeper epistemological issue here, if in the assertion of any truth we must use representations in two different ways. Iconic representations that picture help us refer, as do proper names and indexicals; and analysis is often carried out symbolically, since conditions of intelligibility are thought to hold of certain kinds of things universally. But a representation doesn't wear its function on its face; careful reading of the text in which it occurs is required, in order to see how it should be understood. Pictures sometimes function symbolically, and sometimes they combine an iconic function with a symbolic, analytic one; and they are always subject to conventions of style. Sometimes the thing under investigation can only be given or encountered indirectly, and in consequence of an analysis of its conditions of intelligibility; then its indication by language comes last and is often highly symbolic. We find this often in mathematics when highly infinitary items are investigated. And symbols may often take on an iconic role when they designate newly introduced, abstract objects, like polynomials.

II. Representing, Doing Things with Words, Intervening

The strongly held view that true knowledge ought to be expressed in a single, preferred, univocal idiom, and that subject and predicate terms have the same status, has made the polysemy and ambiguity that I have just claimed is required of veridical language difficult for certain philosophers to see. This is especially striking in philosophy of mathematics, where during the twentieth century some of the most influential philosophers thought that in fact they had invented such an ideal language. I claim, however, that mathematics employs a number of formal modes of representation, which with the help of natural language may be used in tandem or used ambiguously to carry out different linguistic functions. If we were limited to only one, symbolic, axiomatized language for the expression of mathematical truth, we could not do mathematics; but in fact we are not so limited. Mathematicians are able to solve problems successfully because they can tether their polyvalent discourse to mathematical things in many ways; how this happens provides important clues for going beyond "naturalized epistemology" in order to find a theory of knowledge that will work properly for mathematics.

In his book *Representing and Intervening*, Ian Hacking writes, "Science is said to have two aims: theory and experiment. Theories try to say how the world is. Experiment and subsequent technology change the world. We represent and we intervene. We represent in order to intervene, and we intervene in the light of representations. Most of today's debate about scientific realism is couched in terms of theory, representation, and truth." [6] Having made the distinction, Hacking urges the importance of experiment and causal intervention for scientific rationality.

If the terms representing and intervening are severed, we get two incompatible and unsatisfactory views of science. I give them here in caricature to make my point. On the one hand, we have the view of Rudolf Carnap. Scientific rationality is representation. Nature is as it is and we try to describe it truly, in a transparent and univocal language donated by logic to philosophy of science; the true description will be an axiomatized theory, where the first principles are related deductively and inductively to observation statements that report phenomena in the lab and field.[7] On the other hand, we have the view of Nancy Cartwright. Scientific rationality is intervention. We set up artificial environments as nomological machines, and something happens, causally; in doing so we change nature. There is moreover no point in pretending that these local effects can be generalized and described truly by a theory whose first principles are universal principles that describe what must happen in all times and all places.[8]

In his famous essay "Mathematical Truth," Paul Benacerraf uses a version of this disjunction to show that the enterprise of philosophy of mathematics is hopeless.[9] If mathematical rationality is representation, then the vehicle of truth (qua derivability) is the axiomatized theory written in the transparent, univocal language donated by logic to philosophy of mathematics. The problem then is that we cannot designate what we are talking about, since any non-trivial first-order theory has an infinity of models. The instrument of designation would be causal procedures, like those employed in experiments; unfortunately, our access to mathematical entities is not causal. Ian Hacking points out that the hand-wringing occasioned by Thomas Kuhn's book *The Structure of Scientific Revolutions* isn't necessary; incommensurability need not lead directly to irrationalism. I'd make the same observation about the hand-wringing that followed upon Benacerraf's essay. We only have to give up hope for a cogent philosophy of mathematics if we cling to a logicist view of representation and a causalist account of intervention, and moreover forget to look for the middle ground between representation and intervention.

Hacking is guilty of neglecting this middle ground in *Representing and Intervening*, so that his ultimate position is an odd pastiche of "entity realism" and skepticism about "theory realism." In order to profit from his important insights in that book and to transfer them to philosophy of mathematics, I correct his semanticist position by a dose of contemporary pragmatism, and point out some important examples of knowing that occupy the middle ground between representing and intervening. Hacking forgets that language itself can alter what it is about, in a way rather different from the way in which laboratory set-ups and instruments alter what they investigate. One such example is Austin's performative utterance. Another is Ursula Klein's notion of paper tools, illustrated by the development of Berzelian formulas in her book *Experiments, Models, Paper Tools: Cultures of Organic Chemistry in the Nineteenth Century*. [10] Another is the way in which mathematical discourse can be hypostatized to precipitate new items that come to stand in determinate relations with other, previously available items, like polynomials or the well-formed formulas of mathematical logic. The effectiveness of performative utterances, paper tools and hypostatized elements of discourse cannot be explained in material-causal terms, but it does show that language itself (in its many modes) intervenes, constructs, and creates. Scientific rationality understood as a spectrum that includes representation, a middle ground, and intervention, is a clue to a better philosophy of mathematics. Mathematicians represent, construct and intervene (in a semi-causal way, to be explained) in mathematical reality, as chemists represent nature, construct models on paper and in the lab, and create new molecules.

The chemical analogy can be pursued further. As chemists require ambiguous modes of representation to bring chemistry into rational relation with biology on the one hand and physics on the other, and to move between the molecular level and the macroscopic level of the lab, so mathematicians require ambiguous modes of representation to bring different domains into rational relation in order to solve problems, and to move between the finitary and the infinitary. Once we recognize the broad spectrum of ways in which people interact with nature and employ cultural artifacts (including language) broadly construed as modes of representation-to-intervention, we can discern a positive role for ambiguity in language; and this holds as well for the way in which people interact with the things of mathematics. We can be realist about them, and still critical of the truth of any given theory concerning them, and still willing to admit that some mathematical items are creations precipitated by notation and theory, like some molecules.

Here is another way to put my point. I claim that in order to account for the ability of chemistry to refer and describe and construct and intervene, we must look at the manifold uses of language chemists employ and their ability to exploit the ambiguity of some of those modes.[11] To say something true about the energy levels of the benzene molecule, for example, a chemist must use (inter alia) geometric shape, various differential equations, parts of group theory and representation theory, character tables, and the causal record of certain measurements of the behavior of large quantities of benzene molecules, carefully segregated from other kinds of molecules and subject to certain procedures. These representations-to-interventions, juxtaposed and superimposed, must also sometimes be ambiguous in order to allow for meaningful relations between the microscopic and the macroscopic, and between chemical and physical discourse.

The same holds true of mathematicians. To say something true in number theory, for example in the problem context resulting from Andrew Wiles' proof of Fermat's Last Theorem, a mathematician must deploy (inter alia) parts of group theory and representation theory, deformation theory, complex analysis, Arabic notation for the integers and decimal notation for the reals, novel notation for novel algebras, simple geometric forms as the template for certain kinds of diagrams as mappings, and the quasi-causal creation of new items from novel notation; and these modes of representation-to-intervention must allow for meaningful relations between the infinitary and the finite. Indeed, the mathematician's ability to solve problems by profitably relating the infinitary and the finite, or the realms of number and algebraic geometry and complex analysis, *entails* that some of the modes be ambiguous.

III. The First Pages of Singer & Thorpe

In this section and the next, I illustrate these claims by two examples drawn from the algebraic topology textbook *Lecture Notes on Elementary Topology and Geometry* by I. M. Singer and John Thorpe. This influential mid-twentieth century textbook, worked out in the classroom at MIT and Haverford College, was published in 1967. [12] In the first few pages of this book, the authors introduce the fundamental notions of sets, real numbers, and the Euclidean plane. By using a second widely used textbook as a foil, I show that the authors must use a consortium of modes of representation in order to fix the reference of items that will be centrally important to their exposition and to carry out an initial, indispensable analysis of those items.

At the beginning of *Lecture Notes on Elementary Topology and Geometry*, Singer and Thorpe introduce ‘Naive Set Theory’ in the first section of Chapter One, and ‘Topological Spaces’ in the second section. After defining ‘set,’ ‘belonging to,’ ‘subset,’ ‘set equality,’ ‘union,’ ‘intersection’ and ‘complement,’ they define the Cartesian product A x B of two sets A and B as the set of ordered pairs, A x B = [(a,b); a ε A, b ε B], and add that a relation between A and B is a subset R of A x B. They then illustrate this definition by an example.

Example. Let A = B = the set of real numbers. Then A x B is the plane. The order relation x < y is a relation between A and B. This relation is the shaded set of points in Fig. 1.1.

Figure 1.1 is reproduced in Figure 1. Why did Singer and Thorpe add this diagram?

Although the choice of example, and the addition of the diagram seems rather casual, I will argue that both are essential to the project of the textbook, and that the need for the diagram is philosophically important.

There is a conceptual gap between the Cartesian product of the set of real numbers with itself and the plane (presumably they mean the Euclidean plane) that must be bridged in order for the exposition to proceed. The brisk identification is actually a conjurer’s sleight of hand, and Singer and Thorpe were surely aware of this; perhaps this is why they call their first section *naive* set theory. Conversely, Singer and Thorpe could not have proceeded by invoking the Euclidean plane and merely presenting a diagram of it. Analysis, and the topology that springs from it, require the articulation of the line by the real numbers and of the plane by ordered pairs of real numbers referred to a coordinate system of two orthogonal lines, the center that is their point of intersection, and a unit chosen by convention.

My philosophical claim is that the authors of a topology textbook must indicate in particular *what* is referred to, including canonical objects as well as more esoteric objects, and in general *how* those things will be investigated, including a repertoire of methods and results. In order to do this successfully, they must employ a variety of modes of representation whose rational relation is explained in natural language, for two reasons. First, referring to canonical objects and conducting a broad investigation of the conditions of intelligibility of a domain are two quite different functions of representation; one and the same mode of representation used univocally cannot fulfill both roles at once. Second, the canonical object invoked above, the real number line, is a hybrid; its very definition arises at the intersection of different mathematical enterprises with distinct traditions of representation that are transformed but retained in the hybrid. The reconstruction of the Euclidean plane in terms of the Cartesian product of the set of real numbers with itself further complicates the issue. One and the same mode of representation used univocally cannot present the real number line, or the reconstructed plane.

William Boothby, in *An Introduction to Differentiable Manifolds and Riemannian Geometry* [13] reminds us that the product of two (or, more generally, *n*) copies of the set of real numbers with itself is just a set of ordered pairs (or, more generally, *n*-tuples) of real numbers. This means that $\mathbf{R}^2$ (or, more generally, $\mathbf{R}^n$) stands for a whole spectrum of possible mathematical items: vector spaces, metric spaces, topological spaces, and finally (but with some qualification) Euclidean space. Boothby comments, "We must usually decide from context which one is intended." [14] If we decide to treat $\mathbf{R}^n$ as a n-dimensional vector space over $\mathbf{R}$, we must recall that there are many possible n-dimensional vector spaces over $\mathbf{R}$. Though they are all isomorphic, the isomorphism depends on choices of bases in the spaces to be identified by isomorphism, and there is in general no natural or canonical isomorphism independent of these choices. There is, however, one *n*-dimensional vector space over **R** that has a distinguished or canonical basis: this is the vector space of *n*-tuples of real numbers by component-wise addition and scalar multiplication, $\mathbf{R}^n = \mathbf{V}^n$, for which the *n*-tuples $\mathbf{e}_1$ = (1, 0, 0, ... 0), $\mathbf{e}_2$ = (0, 1, 0, ... 0), ..., $\mathbf{e}_n$ = (0, 0, ..., 0, 1) provide a natural or canonical basis.

This choice allows us to restrict what we mean by $\mathbf{R}^n = \mathbf{V}^n$ as a vector space of dimension *n* over **R**, but there is a further choice to be made. An abstract *n*-dimensional vector space over **R** is called Euclidean if it has a positive definite inner product defined on it. Once again, in general there is no natural or canonical way to choose such an inner product, but in the case of $\mathbf{R}^n = \mathbf{V}^n$ a natural inner product is available: $(\mathbf{x}, \mathbf{y}) = \Sigma_{i=1}^{n} x_i y_i$ (where **x** and **y** are vectors and the x_i and y_i are the vector components). Its canonicity is tied to that of the canonical basis for $\mathbf{R}^n = \mathbf{V}^n$ because relative to this inner product the natural basis is orthonormal, so that $(e_i, e_j) = (\delta_{ij})$.

This inner product on $\mathbf{R}^n = \mathbf{V}^n$ (the vector space) can be used to define a metric on $\mathbf{R}^n$ (the Cartesian product of n sets of the real numbers) by first defining the norm of the vector **x** by $\| \mathbf{x} \| = ((\mathbf{x}, \mathbf{x}))^{½}$ and then defining a distance function in terms of it, d (x, y) = $\| \mathbf{x} - \mathbf{y} \|$. (Here x means ($x_1$, x_2,..., x_n) and y means (y_1, y_2, ..., y_n) and (x,y) is a point in $\mathbf{R}^n$). In particular, d (x, 0) = || **x** ||, that is, the norm of the vector **x** is just the distance of the point (x, 0) from the origin, and the definition of the distance of a point from the origin equates vectors (from $\mathbf{R^n} = \mathbf{V^n}$ considered as a vector space) with points (from $\mathbf{R^n}$ considered as the Cartesian product of sets of real numbers), conferring on $\mathbf{R^n}$ the structure of a metric space. Boothby observes, "This notation is frequently useful even when we are dealing with $\mathbf{R^n}$ as a metric space and not using its vector space structure." [15] Having once conferred this metric structure on $\mathbf{R^n}$, we can define "natural or canonical" neighborhoods that are "open balls" around each point of $\mathbf{R^n}$ and use them as the basis for a topology: then $\mathbf{R^n}$ is not only a metric space but a topological space.

Thus we see that depending on the context of use we can mean by $\mathbf{R^n}$ a product of sets, a vector space, a metric space or a topological space. Sometimes (as in the first pages of Singer & Thorpe) it is identified with Euclidean space, and indeed that ultimate identification seems to provide, retro-fittingly, the "natural or canonical" choices of structure that distinguish $\mathbf{R^n}$ as discussed in the preceding paragraphs from all other possible products of sets of real numbers with super-added vector space, metric, and / or topological structure. Euclidean space is the "intended model" for $\mathbf{R^n}$: in particular, the Euclidean line is the "intended model" for **R**, and the

Euclidean plane is the “intended model” for $\mathbf{R}^2$. But even after we have conferred the canonical vector space structure, metric structure and topological structure just discussed, the Cartesian product of sets of real numbers $\mathbf{R}^n$ still isn’t exactly Euclidean space. Boothby writes, “many texts refer to $\mathbf{R}^n$ with the metric d (x, y) as Euclidean space. This identification is misleading in the same sense that it would be misleading to identify *all n*-dimensional vector spaces [over $\mathbf{R}$] with $\mathbf{R}^n$; moreover, unless clearly understood, it is an identification that can hamper clarification of the concept of manifold and the role of coordinates.” [16]

At best, Boothby observes, we can say that $\mathbf{R}^2$ may be identified with $\mathbf{E}^2$ (or, more generally, $\mathbf{R}^n$ may be identified with $\mathbf{E}^n$) *plus a coordinate system*. And there is no “natural or canonical” coordinate system: an arbitrary choice of coordinates is involved, because there is no natural geometrically determined way to identify the two spaces. Having chosen and imposed an origin, a unit, and two mutually perpendicular axes on the Euclidean plane, we can define a one-to-one mapping from the Euclidean plane to $\mathbf{R}^2$ by p –> (x (p), y (p)), the coordinates of p; this mapping is an isometry, preserving distances between points on the Euclidean plane and their images in $\mathbf{R}^2$. This mapping has limitations, however. We can find an analogue for lines on the Euclidean plane in a straightforward way: subsets of $\mathbf{R}^2$ consisting of the solutions of linear equations. Similarly, as Fermat and Descartes discovered, subsets of $\mathbf{R}^2$ consisting of the solutions of quadratic equations correspond in a straightforward way to circles and conic sections.

However, some common and canonical entities on the Euclidean plane do not correspond to easily definable subsets of $\mathbf{R}^2$: the best example is the triangle and more generally plane polygons. One must use a cut-and-paste method that is formally un-illuminating. Moreover, the imposition of a coordinate system on the Euclidean plane generates "artifacts" that are geometrically insignificant, the way that staining a section of a cell for an electron microscope slide generates biologically insignificant artifacts (even as it exhibits other, biologically significant items and properties). Boothby's example of this is especially interesting, because he shows that to define a geometrically significant thing (the angle between two lines) we must make an excursion into the geometrically insignificant. Given a coordinate system on the Euclidean plane, and the identification of a line with the graph of a linear equation, we define its slope and its *y*-intercept b when we write: L = $\{ (x,y) \mid y = \mathrm{m}x + \mathrm{b} \}$. The values of m and b are not geometrically significant, since they arise simply from the choice of coordinate system. However, given two lines thus represented, the value $(\mathrm{m}_2 - \mathrm{m}_1 / 1 + \mathrm{m}_1\mathrm{m}_2)$

does have geometrical meaning: its value, the tangent of the angle between the lines, is independent of the choice of coordinates.

In sum, the relation between the Euclidean plane and the Cartesian product of the set of real numbers is an analogy, which may be made determinate by various kinds of representations in the service of solving various kinds of problems. The notion of isomorphism central to the semantic approach in the philosophy of science is here insufficient. First, the choice of the most meaningful or best isomorphism cannot be explained without reference to the pragmatic context of problem solving: with respect to what mathematical intent are we representing the Euclidean plane? Second, we cannot explain the "natural or canonical" status of the Euclidean plane in the sea of isomorphic (or homomorphic or homeomorphic or isometric) versions of the Cartesian product of the set of real numbers. Thus, Singer and Thorpe *must* add a diagram of the Cartesian plane to their definitions in the first section of the first chapter of their topology book. The symbolic and much too general notation $\mathbf{R}^2$ must be supplemented by an icon, in this case a picture of the plane nicely fitted out with a coordinate system and shading that indicates its indefinite / infinite extent, as well as natural language to bring the two kinds of representation into rational relation: "Let A = B = the set of real numbers. Then A x B is the plane." The same kind of argument could be made for use of diagrams in Figure 1.2 of the second section (Figure 2), where the definition of metric space precedes and motivates the definition of a topological space.

The Euclidean plane and the Euclidean line play the role of canonical object here, as they do throughout analysis and throughout differential geometry. For a function to be differentiated and integrated, for a manifold to be tractable, it must be locally linearizable – which means to stand locally in a special relation to Euclidean *n*-dimensional space – and then that condition must somehow be made global. Boothby, having completed his account of the gap between Euclidean geometry and $\mathbf{R}^{\mathbf{n}}$, concludes, "We need to develop both the coordinate method and the coordinate-free method of approach. Thus we shall often seek ways of looking at manifolds and their geometry which do not involve coordinates, but will use coordinates as a useful computational device." [17] This means that we cannot avoid the use of diagrams like Figure 1 and Figure 2, but it does not, I think, mean only that, because Boothby is speaking not just of the line and the plane, but of *n*-dimensional Euclidean geometry generally.

It certainly follows from what he says about the pedagogical approach of his textbook that we must use a variety of modes of representation to do (in this case) differential geometry properly, to know what we are talking about and to find effective methods for solving problems about them. Boothby invokes the study of geometry by the Greeks and by high school students (who pose and solve problems about figures without benefit of coordinate systems), just at the point where he makes a dutiful but non-committal gesture towards Hilbert's axiomatization of geometry: "It is very tricky and difficult to give a suitable definition of Euclidean space of any dimension, in the spirit of Euclid, that is, by giving axioms for (abstract) Euclidean space as one does for abstract vector spaces. This difficulty was certainly recognized for a very long time, and has interested many great mathematicians... Careful axiomatic definition of Euclidean space is given by Hilbert. Since our use of Euclidean geometry is mainly to aid our intuition, we shall be content with assuming that the reader "knows" this geometry from high school." [18] Though Boothby uses the term 'intuition,' he might more accurately have invoked the notion of mathematical experience, which teaches us the strengths and limitations of various modes of representation, employed singly or in tandem, as they help us identify and analyze the things of mathematics.

IV. De Rham's Theorem

In Chapter 4 of Singer and Thorpe the authors introduce simplicial complexes; in Chapter 5 they introduce smooth manifolds; and in Chapter 6 they prove De Rham's theorem, which links simplicial complexes and smooth manifolds in a significant way by means of group theory. [19] In this problem content, abstract algebra provides representations for topology that are effective *when combined with* other sorts of representations, including icons and natural language. What does effective at problem-solving mean in this context? A family of problems that occurs naturally with respect to manifolds is how to understand their global structure in light of their local structure. One version of this is how to classify certain kinds of manifolds. Another version is how to carry out real or complex analysis on a manifold, how to define functions and integrate and differentiate functions on them; and more generally how to understand what happens when real or complex analysis is transferred to manifolds, and how that transformation of analysis on flat spaces relates to the geometry of the manifolds. Mathematicians often start by trying out new methods on well understood, canonical objects and then extending them to others, less well understood or harder to represent, for example, higher dimensional manifolds. In algebraic topology these methods include associating topological structures (and maps between them) with group structure (and maps between them), and infinitary items and groups with finite, combinatorial items and groups.

In Chapters 4-6 of Singer & Thorpe, the sphere and the torus, among others, play the role of canonical objects. They are first of all geometrical objects with straightforward icons that reflect certain conventions about how to represent three dimensions on a 2-dimensional page. (Figures 3 and 4) [20] As a variety, a given sphere or torus can be represented by a system of polynomial equations $f_1(x_1, \ldots x_n) = 0$,… $f_n(x_1, \ldots x_n) = 0$, because it is the set of points on which the polynomials vanish. As a manifold, the sphere or torus is viewed as a certain kind of topological space. This representation presupposes that each one may be decomposed into a set of points S and that we may choose some subset of all the subsets of S, $T \quad 2^S$ satisfying certain conditions: the empty set and S itself are in T, all finite intersections of members of T are again in T and so are all arbitrary unions. Then we call T a collection of “open sets” on S and the pair (S, T) is a topological space. (To suppose that any geometrical entity may be treated as a set of points or elements and that we can intelligibly locate its power set requires further reflection and justification.) We can give any set what is called the discrete topology by stipulating that $T = 2^S$, the power set of S; this is why I said earlier that any set can be thought of as a topological space. A final note:

what corresponds to an isomorphism in topology is a homeomorphism. A function $f: S \rightarrow T$ is continuous if the inverse images of open sets are open; f is a homeomorphism if f is a one-one correspondence and both f and its inverse are continuous.

Our sphere and torus have now been represented by an icon; a system of polynomial equations; a highly infinitary (who knows how high?) set of points; and a topological space. There is more to come, but we now need to make a digression to explain the definition of a topological space. In analytic geometry, we are used to knowing the distance between two points, and being able to assign a number to it: this means we are regarding our space as a kind of metric space S. It is defined as a set S together with a function ρ: $S \times S \rightarrow \mathbf{R}^+$ (the non-negative reals) such that for all s_1, s_2 and s_3 in S,

$\rho(s_1, s_2) = 0$ if and only if $s_1 = s_2$

$\rho(s_1, s_2) = \rho(s_2, s_1)$

$\rho(s_1, s_3) = \rho(s_1, s_2) + \rho(s_2, s_3)$

This abstract definition of distance as a metric ρ was first given by Fréchet in 1906, but despite its generalization of the notion of distance, it is too special in certain respects. First, different metrics may end up furnishing the same notion of “openness” or of neighborhoods for e.g. Euclidean space; thus mathematicians were led to formulate the definition of a topological space as a way of defining a more abstract structure that captures “openness,” neighborhoods, limit points and continuity. (A closed set in topology is the complement of an open set; the closure of an open set is that set with all its limit points.) Second, some mathematical entities cannot be given a natural metric, but can be given a useful topological structure.

The definition of a topological space raises the question whether a given topological space is rich (but not too rich—the discrete topology is uninformative) in open sets. If there are enough open sets to separate points, the space is called Hausdoff; if there are enough open sets to separate closed sets, the space is called normal. A space with a metric, S, can be turned into a topological space by taking as the open sets unions of balls, where a ball of radius *a* (*a* real) about $s_0 \varepsilon S$ is defined as the set, $[s \varepsilon S \mid \rho (s, s_0) < a]$. If the metric is the usual Pythagorean metric, the ball is the interior of a circle or sphere or n-sphere of radius *a*. All metric spaces turned into topological spaces this way are Hausdorff and normal. Now we can turn to the representation of our sphere and torus as manifolds. A manifold is a Hausdorff space with additional structure, a set of maps Φ such that for each

$s \varepsilon S$ there is a $\varphi_s \varepsilon \Phi$ that maps some open set containing s homeomorphically into an open set in $\mathbf{R}^2$.

Thinking of a surface as a manifold leads to thinking of it as articulated into cells or panes or the faces of a triangulation, the bits that are flat “enough” to be mapped nicely to 2-dimensional Euclidean space, the plane. A manifold can be linearized locally: the problem then is to move from the local reduction to a global reduction, so as to extend some version of the nice properties that follow from linearization to the manifold as a whole. In order to get the maps to overlap with each other in a way that accommodates this extension, we must add further conditions governing what happens on the overlap—the nature of the maps in Φ. The manifold is called a C^0-manifold if the mappings overlap in a way that is continuous; C^k if all partial derivatives of order $\leq k$ exist and are continuous; C^∞ if all partial derivatives of all orders exist and are continuous; and C^ω if it is real analytic.

We represent our sphere and torus as C^∞ manifolds, “smooth manifolds.” Because smooth manifolds are so well-behaved, they have a tangent space (and a cotangent space) at every point and all the tangent spaces (and cotangent spaces) can be collected into tangent (and cotangent) bundles that allow us to define vector fields and differential 1-forms on the manifold. A construction called a Grassmann algebra or exterior algebra allows us to define the set of smooth k-forms on the manifold, and then the set of all smooth differential forms on the manifold. A smooth differential form ω on a smooth manifold X is *closed* if $d\omega = 0$; it is exact if it is the differential of another form on X, that is, if $\omega = d\tau$ for some τ. Every exact form is closed, since $d^2 = 0$. We are now way out there in the universe of abstraction, but the path back to earth is surprisingly rapid; in the case at hand, it is really surprising because of the connections forged between the local properties and the global structure of the space X.

The construction of homology and cohomology groups depends on a continuous map, a 'boundary operator,' in this case 'd', such that when the map is applied twice, the result is a 'zero' in the space. Call $Z^k(X,d)$ the vector space of closed k-forms on X, and $B^k(X,d)$ the space of exact k-forms on X; then $H^k(X,d)$ is defined as $Z^k(X,d) / B^k(X,d)$ and it is the k^{th} De Rham cohomology group of X. Even though $Z^k(X,d)$ and $B^k(X,d)$ may be highly infinitary, the dimension of their quotient $H^k(X,d)$ is finite for a compact manifold and is called the k^{th} Betti number of X. The dimension of group $H^0(X,d)$ where X is a smooth manifold measures the number of connected components of the manifold and the dimension of group $H^1(X,d)$ measures the number of 'holes.' Thus, the former group assigns the same number to both the sphere and the torus, and the latter group assigns different numbers to them. Now we see the sphere or the torus represented by tangent and cotangent bundles, sets of smooth differential forms borrowed from analysis with a complicated algebraic structure (the Grassman algebra), quotients of k-forms that may turn out to be groups of finite dimension, and, finally, integers that count components and holes of the original manifold.

These representations work in tandem to pull up and away from the sphere and the torus into a highly abstract realm, which in turn precipitates us back down to the realm of whole numbers. Though cohomology groups are defined in terms of the manifold structure of X, they are topological invariants: if two manifolds are homeomorphic, then they have isomorphic cohomology groups. Indeed these groups can be defined directly using only the topological structure of the manifold. Moreover, each smooth manifold X (here the sphere or the torus) has been associated with a new algebraic invariant, the group $H^k(X,d)$, so that a smooth map between manifolds induces algebraic maps between these algebraic objects. Thus, difficult topological problems can be approached and sometimes solved by studying homology groups and crunching numbers.

Another way of linearizing a manifold does so globally by bringing the whole thing into relation with a simplicial complex, though the "bringing into relation" depends on the locally Euclidean structure, smoothness, and compactness of the manifold. This is a modern, abstract version of Archimedes' method of approximating curves by inscribed and circumscribed polygonal lines. A simplicial complex is most straightforwardly, though imprecisely, described as a concatenation of 'simplexes' of different dimensions: points, line segments, triangles, tetrahedrons, and so forth. Two simplices that compose a simplicial complex must always intersect in a simplex; and all components of a simplex must also belong to the simplicial complex to which that simplex belongs. The "triangulation" of a surface that is a smooth manifold, for example, is carried out by finding a homeomorphism from an appropriate simplicial complex to the manifold, whose restriction to each simplex is a nice map satisfying certain conditions. Every compact, smooth manifold can be smoothly triangulated. The obvious advantage of simplicial complexes is that they can be treated as a finite set of numbers: the number of their vertices, edges, faces, and so forth. They are combinatorial items.

Simplicial complexes, like manifolds, can also be represented by homology and cohomology groups, defined in terms of a boundary operator

∂ satisfying the condition $\partial^2 = 0$. Every simplicial complex K can be used to define a group C_l (K, G) (G an arbitrary abelian group) using the l-dimensional (oriented) simplices of a simplicial complex in a certain construction called l-chains. The boundary operator ∂ maps n+1 dimensional simplices to their n-dimensional boundaries in a certain way, and induces a map from C_{l+1} (K, G) $\rightarrow$ C_l (K, G) $\rightarrow$ C_{l-1} (K, G). (When G is a field F, C_l (K, F) is a vector space over F whose dimension equals the number of l-simplices of K.) The group Z_l (K, G) consists of all the l-chains that are mapped to 0 by ∂ (called cycles); B_l (K, G) consists of all the l-chains that are obtained by ∂ operating on some l+1 cycle (called boundaries); and H_l (K, G) is the quotient group Z_l (K, G) / B_l (K, G). It is the l^{th} homology group of K with coefficients in G. It turns out that the groups H_l (K, G) depend only on the topology of [K], where [K] is the point set union of the open simplices of K, so that a homeomorphism from [K] to [L] induces appropriate isomorphisms between the associated homology groups. When G = **R** (the reals), the group H_l (K, **R**) is also a vector space over **R** whose dimension is called the l^{th} Betti number β_l of K, and the Euler characteristic of K, χ (K), is equal to the sum Σ $(-1)^l$ β_l where l ranges from 0 to the dimension of K. From these stipulations, it follows that the Euler characteristic of K also equals the sum Σ $(-1)^l$ α_l (where l ranges from 0 to the dimension of K) in which the α_l denote the number of l-simplices in K; that is, the Euler number of K is equal to the number of vertices – the number of edges + the number of 2-faces – et cetera.

This brings us down to earth again, to the sphere and the torus, though this time we only have to descend from the lower stratosphere of combinatorics. If [K] is homeomorphic to a connected, compact, orientable 2-dimensional manifold, then $\beta_0 = 1$ and $\beta_2 = 1$, so that χ (K) = 2 – β_1 and β_1 = 2 - χ (K). Passing via simplicial complexes, it can be shown that any such surface is homeomorphic to a sphere with a certain number of handles and ½ β_1 (β_1 is always even) gives the number of handles. Clearly, a torus is homeomorphic to a sphere with one handle. In sum, the homology groups completely determine the homeomorphism class of connected, compact, orientable surfaces. However, Singer and Thorpe remark, for higher dimensional manifolds (and for surfaces that fail to be connected, compact, or orientable) the homology groups are not so informative.

One way to exhibit a homeomorphism between a smooth manifold X and a simplicial complex K is to find a map h which, when restricted to each simplex of K, maps nicely to a smooth submanifold of X. The triple (X, K, h) is called a “triangulation” of X, and it can be then be used to link the differential operator d and smooth l-forms on the manifold to the boundary operator ∂ and oriented (l+1)-simplexes of the associated simplicial complex in a version of Stokes' Theorem. [21] (To do this requires the technical trick of passing from the homology theory of simplicial complexes to its dual cohomology theory.) This sets up the proof of De Rham's Theorem, which asserts that if (X, K, h) is a smoothly triangulated manifold, then the map from the cohomology group of X to the cohomology group of K is an isomorphism for each l between 0 and the dimension of X. It is important to see the intentionality built into this homeomorphism and the group isomorphism that goes along with it. The theorem depends on homology groups that isolate focal objects, helping to pinpoint the structure the mathematician is interested in, while at the same time forging the link between local information and global information. Moreover, the simplicial complexes are studied because they contribute to an understanding of manifolds, and not vice versa, even though De Rham's theorem is expressed in terms of an isomorphism. [22] Because simplicial complexes are combinatorial items, the process of triangulation in concert with De Rham's theorem makes some aspects of some manifolds a matter of numerical computation.

We should notice how hard the topologists had to work to arrive at the series of group isomorphisms demonstrated by De Rham’s Theorem. The analysis of well-understood canonical objects deploys a variety of representations, in which reductions link the representations, and novel constructions elaborate those linkages into groups and algebras. The constructive and reductive procedures in turn often invoke, or assume as accessible and well-understood, further canonical objects: the Euclidean plane, the real number line, the circle, the triangle. The process involves many different kinds of mappings, and shifts the focus of its interest from one set of related objects to another, depending on the success and interest of the mappings. Isomorphisms emerge from this process, but they are highly constrained when they do emerge, and what they mean depends on an understanding of the problems they are designed to solve, and the most important objects that figure in those problems.

V. A Concluding Meditation on Truth

In *The Dappled World: A Study of the Boundaries of Science*, Nancy Cartwright's meditation on why we must reflect more deeply on the relationship between the abstract and the concrete bears directly on the point I have just been developing. She uses her argument to talk about episodes in modern physics, yet clearly what she says has a more general import for epistemology. Chapter 2 of her book recalls Lessing's distinction between intuitive cognition, where we attend to our ideas of things (I would say, to our awareness of things) and symbolic cognition, where we attend to signs we have substituted for them. She writes, "All universal and general knowledge, all systematic knowledge, requires symbolic cognition; yet only intuitive cognition can guarantee truth and certainty. For words are arbitrary, you cannot see through them to the idea behind them," and quotes Lessing: "Intuitive knowledge is clear in itself; the symbolic borrows its clarity from the intuitive… In order to give a general symbolic conclusion all the clarity of which it is capable, that is, in order to elucidate it as much as possible, we must reduce it to the particular in order to know it intuitively."[23] While I contest Lessing's Cartesian and Kantian assumptions about "intuitive cognition" as given, self-evident, and accessible independent of language, I agree with him (and Cartwright, and Hendry) that in order to extend our knowledge we must be able to indicate the ‘what’ that we are aware of, as well as the ‘why’ and ‘how.’

Cartwright’s reading of Lessing discerns here both an epistemological claim (the general only becomes graphic or visible—I would say, intelligible—in the particular) and an ontological claim (the general exists only in the particular). Then she shows that these claims are pertinent to understanding how the theories of physics work. “First, a concept that is abstract relative to another more concrete set of descriptions never applies unless one of the more concrete descriptions also applies. These are the descriptions that can be used to ‘fit out’ the abstract description on any given occasion. Second, satisfying the associated concrete description that applies on a particular occasion is what satisfying the abstract description consists in on that occasion.” [24] In other words, abstract descriptions (like homology and cohomology theory) can only be used to say true things if they are combined with concrete descriptions that fix their reference in any given situation. This insight holds for mathematics as well as for physics. When abstract schemata are applied in mathematics, their successful application depends on the mathematician’s ability to find a useful concrete description for the occasion, which will mediate between complex mathematical reality and the general theory.

This does not, however, entail that abstract concepts are no more than collections of concrete concepts. Cartwright argues, "The meaning of an abstract concept depends to a large extent on its relations to other equally abstract concepts and cannot be given exclusively in terms of the more concrete concepts that fit it out from occasion to occasion." [25] The more abstract description of the situation adds important information that cannot be "unpacked" from any or even all of the concrete descriptions that might supplement it, or from our unreflective awareness of the thing or things successfully denoted. Likewise, the more concrete descriptions have meanings of their own that are to a large extent independent of the meaning of any given abstract term they fall under; we cannot deduce the content of the concrete descriptions by specifying a few parameters or by plugging in constants for variables in the abstract description. The relation between the abstract description and the concrete descriptions is not the same as the Aristotelian relation between genus and species.

Here is another way of putting the insight. Abstract terms can only be used to say something true when they are combined with more concrete locutions in different situations that help us to fix their reference. And there is no “sum” of such concrete locutions. Conversely, as Leibniz argues, concrete terms can only be used to say something true when they are combined with more abstract locutions that express the conditions of intelligibility of the thing denoted, the formal causes that make the thing what it is, and so make its resemblance to other things possible. And there is no “sum” of the conditions of intelligibility of a thing.[26] We cannot totalize concrete terms to produce an abstract term and we cannot totalize abstract terms to produce a concrete term that names a thing; and furthermore we cannot write meaningfully and truthfully without distinguishing as well as combining concrete and abstract terms.

So we are left with an “essential ambiguity” that results from the logical slippage that *must* obtain between the concrete and the abstract. There is an inhomogeneity that cannot be abolished, which obtains between the abstract terms that exhibit and organize the intelligibility of things, and the concrete terms that exhibit how our understanding bears on things that exist in the many ways that things exist. We need to use language that both exhibits the ‘what’ of the discourse, and identifies the formal causes, the ‘why’ of the things investigated. To do so, we need to use different modes of representation in tandem, or to use the same mode of representation in different ways, to use it ambiguously. In this essay, I have tried to show that the pursuit of knowledge about topological spaces proceeds in this way. Set theory, real analysis and abstract algebra provide modes of representation for topology that are effective when combined with other sorts of representations, including icons and natural language. And this is true for fundamental definitions that must be given at the beginning of any investigation of algebraic topology, as well as of particularly important theorems.

Footnotes

[1] I would like to thank the National Endowment for the Humanties for awarding me a Fellowship in 2004 that supported the research for this essay, and the Pennsylvania State University for a sabbatical year in Paris during 2004-5. I would also like to thank Dr. Karine Chemla, Director of REHSEIS (Equipe Recherches Epistémologiques et Historiques sur les Sciences Exactes et les Institutions Scientifiques), Université Paris 7 et CNRS (Centre National de la Recherche Scientifique) and her colleagues, who invited me to work with them during my sabbatical.

[2] Hendry 2001, 227.

[3] Grosholz and Yakira 1998, Ch. 1.

[4] Peirce distinguishes between icons, which resemble what they represent, and symbols, which represent only by convention. Peirce 1998, 226-241.

[5] Grosholz and Yakira 1998, Ch. 1.

[6] Hacking 1983, 31.

[7] Carnap 1928 / 1967.

[8] Cartwright 1999.

[9] Benacerraf 1973.

[10] Klein 2003. See my review – Klein 2005.

[11] Grosholz and Hoffmann 2000, 230-247.

[12] Singer and Thorpe 1967.

[13] Boothby 1975.

[14] Ibid, 2.

[15] Ibid, 3.

[16] Ibid., 4.

[17] Ibid., 5.

[18] Ibid., 4.

[19] Singer and Thorpe 1967, 69-156.

[20] Ibid., 1967, 135 and 143.

[21] Stokes Theorem says that if M is a smooth manifold of dimension n and ω is a certain nicely behaved kind of *n*-1 differential form on M, and if ∂M denotes the boundary of M with its induced orientation, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

The Stokes theorem can be considered as a generalisation of the fundamental theorem of the calculus in the case where the manifold is **R**, the real line.

[22] My thanks to Joseph Mazur for his insights into the meaning of homology and cohomology in algebraic topology.

[23] Cartwright 1999, 38.

[24] Ibid., 49.

[25] Ibid., 40.

[26] Grosholz and Yakira 1998, 56-72.

Bibliographical References

Benacerraf 1973 P. BENACERRAF, *Mathematical Truth*, in *Journal of Philosophy* 70 / 19 (1973), 661-679.

Boothby 1975 W. BOOTHBY, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, New York 1975.

Carnap 1928 R. CARNAP, *Der Logische Aufbau der Welt*, Berlin 1928.

Carnap 1967 R. CARNAP, *The Logical Structure of the World*, B. ROLF [trans. by], Berkeley 1967.

Cartwright 1999 N. CARTWRIGHT, *The Dappled World*, Cambridge 1999.

Grosholz, E. and Yakira, E. 1998 E. GROSHOLZ and E. YAKIRA, *Leibniz's Science of the Rational*, *Studia Leibnitiana* Sonderheft 26, Stuttgart 1998.

Grosholz, E. and Hoffmann, R. 2000 E. GROSHOLZ and R. HOFFMANN, *How Symbolic and Iconic Languages Bridge the Two Worlds of the Chemist: A Case Study from Contemporary Bioorganic Chemistry*, in N. Bhushan and S. Rosenfeld [ed. by], *Of*

Minds and Molecules: New Philosophical Perspectives on Chemistry, Oxford 2000, 230-247.

Hacking 1983 — I. HACKING, *Representing and Intervening*, Cambridge 1983.

Hendry 2001: — R. HENDRY, *Mathematics, Representation and Molecular Structure*, in U. KLEIN [ed. by], *Tools and Modes of Representation in the Laboratory Sciences*, Dordrecht 2001, 221-236.

Klein 2003 — U. KLEIN, *Experiments, Models, Paper Tools: Cultures of Organic Chemistry in the Nineteenth Century*, Stanford 2003.

Klein 2005 — U. KLEIN, *Experiments, Models, Paper Tools: Cultures of Organic Chemistry in the Nineteenth Century*, «Studies in History and Philosophy of Science», 36 (2005), 411- 417.

Peirce 1998. — C. S. PEIRCE, *Pragmatism as the Logic of Abduction*, in *The Essential Peirce*, Vol. 2, Indianapolis 1998, 226-241.

Singer, I. M. and Thorpe, J. 1967 — I. M. SINGER and J. THORPE, *Lecture Notes on Elementary Topology and Geometry*, Glenville 1967.