Productive Ambiguity in Leibniz's Representation of Infinitesimals

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In his theory of knowledge and mathematics, Leibniz announces not that (as Pope would have it) 'The study of mankind is Man,' but rather that 'The study of mankind is God.' To understand human knowledge and the things of mathematics properly, according to Leibniz, we must first study how these ideas are manifest in God. This claim is underwritten by the Principle of Sufficient Reason in two versions: the Principle of Perfection, the version pertinent to God, which guarantees that the created world stands in rational relation to the realm of possibles; and the Principle of Continuity, the version pertinent to finite humankind, which guarantees that the infinitary stands in rational relation to the finitary. The great advantage of the latter is that it can yoke things that are radically unlike, and bring them into rational relation without abrogating their unlikeness. Moreover, what we create under the aegis of the Principle of Continuity may be understood vis à vis what God creates under the aegis of the Principle of Perfection: thus the study of human creativity also begins with the study of God. We are not only truly free, but truly creative; and our creativity is what makes God=s creation the maximum, indeed the increasingly maximal, perfection that it is.

In this essay, I argue that Leibniz believed that mathematics is best investigated by means of a variety of modes of representation, often stemming from a variety of traditions of research, like our investigations of the natural world and of the moral law. I expound this belief with respect to two of his great metaphysical principles, the Principle of Perfection and the Principle of Continuity, both versions of the Principle of Sufficient Reason; the tension between the latter and the Principle of Contradiction is what keeps Leibniz's metaphysics from triviality. I then illustrate my exposition with two case studies from Leibniz's mathematical research, his development of the infinitesimal calculus, and his investigations of transcendental curves.

1. The Principle of Continuity

Leibniz wrote a public letter to Christian Wolff, written in response to a controversy over the reality of certain mathematical items sparked by Guido Grandi; it was published in the Supplementa to the *Acta Eruditorum* in 1713 under the title 'Epistola ad V. Cl. Christianum Wolfium, Professorem Matheseos Halensem, circa Scientiam Infiniti.'[1] Towards the end, he presents a diagram (discussed below in Section 2) and concludes, 'All this accords with the *Law of Continuity*, which I first proposed in the Nouvelles de la République des Lettres de Bayle et appliqué aux Lois du Mouvement.[2] It entails that with respect to continuous things, one can treat an external extremum as if it were internal [*ut in continuis extremum exclusivum tractari posit ut inclusivum*], so that the last case or instance, even if it is of a nature completely different, is subsumed under the general law governing the others.' He cites as illustration the relation of rest to motion and of the point to the line: rest can be treated as if it were evanescent motion and the point as if it were an evanescent line, an infinitely small line. Indeed, Leibniz gives as another formulation of the Principle of Continuity the claim that 'the equation is an infinitesimally small inequality.' (MS VII 25, for example)

The Principle of Continuity, he notes, is very useful for the art of invention: it brings the fictive and imaginary (in particular, the infinitely small) into rational relation with the real, and allows us to treat them with a kind of rationally motivated tolerance. For Leibniz, the infinitely small cannot be accorded the intelligible reality we attribute to finite mathematical entities because of its indeterminacy; yet it is undeniably a useful tool for engaging the continuum, and continuous items and procedures, mathematically. The Principle of Continuity gives us a way to shepherd the infinitely small, despite its indeterminacy, into the fold of the rational. It is useful in another sense as well: not only geometry but also nature proceeds in a continuous fashion, so the Principle of Continuity guides the development of mathematical mechanics.

But how can we make sense of a rule that holds radically unlike (or, to use Leibniz's word, heterogeneous) terms together in intelligible relation? I want to argue that two conditions are needed. First, Leibniz must preserve and exploit the distinction between ratios and fractions, because the classical notion of ratio presupposes that while ratios link homogeneous things, proportions may hold together inhomogeneous ratios in a relation of analogy that is not an equation. This allowance for heterogeneity disappears with the replacement of ratios by fractions: numerator, denominator, and fraction all become numbers, and the analogy of the proportion collapses into an equation between numbers.[3] However, Leibniz's application of the Principle of Continuity is more strenuous than the mere discernment of analogy: the relation between 3 and 4 is analogous to the relation between the legs of a certain finite right triangle. But the relation between the legs of a finite and those of an infinitesimal 3-4-5 right triangle is not mere analogy; the analogy holds not only because the triangles are similar but also because of the additional assumption that as we allow the 3-4-5 right triangle to become smaller and finally evanescent, 'the last case or instance, even if it is of a nature completely different, is subsumed under the general law governing the others.'

Thus, the notation of proportions must co-exist beside the notation of equations; but even that combination will not be sufficient to express the force of the Principle of Continuity. The expression and application of the principle requires as a second condition the adjunction of geometrical diagrams. They are not, however, Euclidean diagrams, but have been transformed by the Principle of Continuity into productively ambiguous diagrams whose significance is then explicated by algebraic equations, differential equations, proportions, and / or infinite series, and the links among them in turn explicated by natural language. In these diagrams, the configuration can be read as finite or as infinitesimal (and sometimes infinitary), depending on the demands of the argument; and their productive ambiguity, which is not eliminated but made meaningful by its employment in problem-solving, exhibits what it means for a rule to hold radically unlike things together.

This is a pattern of reasoning, constant throughout Leibniz's career as a mathematician, which the Logicists who appropriated Leibniz following Louis Couturat and Bertrand Russell could not discern, much less appreciate. As Herbert Breger argues in his essay 'Weyl, Leibniz und das Kontinuum,' the Principle of Continuity and indeed Leibniz's conception of the continuum—indebted to Aristotle on the one hand, and seminal for Hermann Weyl, Friedrich Kaulbach and G.-G. Granger on the other—is inconsistent with the Logicist program, even the moderate logicism espoused by Leibniz himself not to mention the more radical versions popular in the twentieth century. The intuition [*Anschauung*] of the continuous, as Leibniz understood it, and the methods of his mathematical problem-solving, cannot be subsumed under the aegis of logical identity. Breger adds, 'I can't go into this conjecture here, and would like simply to assert that although Leibniz did advocate a philosophical program corresponding to Logicism, he also distanced himself a great deal from it in his mathematical practice.'[4] In the two sections that complete this essay, I will show that this pattern of reasoning characterizes Leibniz's thinking about, and way of handling, non-finite magnitudes throughout his active life as a mathematician.

2. Studies for the Infinitesimal Calculus

In 1674, Leibniz wrote a draft entitled ‘De la Methode de l’Universalité,’ (C, 97-122; Bodemann V, 10, f, 11-24) in which he examines the use of a combination of algebraic, geometric and arithmetic notations, and defends a striking form of ambiguity in the notations as necessary for the ‘harmonization’ of various mathematical results, once treated separately but now unified by his new method. He discusses two different kinds of ambiguity, the first dealing with signs and the second with letters.

The simplest case he treats is represented this way:

1___________1______1______1

A C B C

The point of the array is to represent a situation where A and B are fixed points on a line; this means that if the line segment AC may be determined by means of the line segment AB and a fixed line segment BC = CB, there is an ambiguity: the point C may logically have two possible locations, one on each side of B. Leibniz proposes to represent this situation by a sole equation, which however involves a new kind of notation. He writes it this way:

AC = AB ╪ BC

and goes on to suggest a series of new operations, corresponding to cases where there are three, four, or more fixed points to begin with. He generates the new symbols for operations by a line underneath (which negates the operation) or by juxtaposing symbols. (One sees some nascent group theory here.) (C, 100)

Re-expressing the same point in algebraic notation, he writes that

╪ $a + b$, or $+ a$ ╪ $b = c$

means that

$+ a + b$, or $- a + b$, or $+ a + b$, or $+ a - b$

and goes on to give a more complex classification for ambiguous signs. The important point, however, is that the ambiguous signs can be written as a finite number of cases involving unambiguous signs. (C, 102)

The treatment of ambiguous letters, however, is more complex, truly ambiguous, and fruitful. He illustrates his point with a bit of smoothly curved line AB(B)C intersected at the two points B and (B) by a bit of straight line DB(B)E. The notation AB(B)C and DB(B)E is ambiguous in two different senses, he observes. On the one hand, the concatenated letters may stand for a line, or they may stand for a number, 'puisque les nombres se representent par les divisions du continu en parties egales,' and because, by implication, Descartes has shown us how to understand products, quotients, and n^{th} roots of line segments as line segments. On the other hand, and this is a second kind of ambiguity, lines may be read as finite, as infinitely large, or as infinitely small. The mathematical context will tell us how to read the diagram, and he offers the diagram just described as an example: '…Thus in order to understand that the ligne DE is the tangent, one has only to imagine that the line B(B) or the distance between the two points where it intersects the curve is infinitely small: and this is sufficient for finding the tangents.'[5]

In this configuration, reading B(B) as finite so that the straight line is a secant, and as infinitesimal so that the straight line is a tangent, is essential to viewing the 'harmony' among the cases, or, to put it another way, to viewing the situation as an application of the Principle of Continuity. The fact that they are all represented by the same configuration, supposing that B(B) may be read as ambiguous, exhibits the important fact that the tangent is a limit case subject to the same structural constraints as the series of secants that approach it. And this is the key to the method of determining tangents. A good characteristic allows us to discern the harmony of cases, which is the key to the discovery of general methods; but such a characteristic must then be ambiguous.

To further develop the point, Leibniz returns to his original example, adumbrated.

A   B

1___________1______1______1______1______1

1C   (3C)   3C   ((3C))   2C

Once again, A and B are fixed points on a line. When we set out the conditions of the problem where a line segment AC is determined by two others, AB and BC, the point C may fall not only to the left or right of B, but directly on B: ‘the point C which is moveable may fall on the point B.’[6] Since we want the equation

AC = +AB ╪ BC

to remain always true, we must be sure to include the case where B and C coincide, that is, where BC is infinitely small, ‘so that the equation may not contradict the equality between AC and AB.’[7] In other words, the equality AC = AB is a limit case of the equation just given. In order to exhibit its status as a limit case, or (to use Leibniz’s vocabulary) to exhibit the harmony among these arithmetic facts and thus the full scope of the equation, we must allow that BC may be infinitely small.

Here, Leibniz observes, the ambiguity of the *sign* ╪ is beside the point and doesn’t matter; but the ambiguity of the *letters* is essential for the application of the principle of continuity, and thus cannot be resolved but must be preserved.

‘Since one may place 3C, not only directly under B, in order to make AC = AB and BC equal to zero, but over towards A at (3C), or over on the other side of B at ((3C)) in order to make the equation AC = + AB – BC true on the one hand or on the other to make the equation AC = + AB + BC true, provided that the line (3C)B or ((3C))B be conceived as infinitely small. You see how this observation can serve the method of universality in order to apply a general formula to a particular case.’[8] In the diagram, (3C) and 3C, or 3C and ((3C)) may be identified when AC = BC, as B and (B) are in the preceding diagram when the secant becomes the tangent.

Leibniz’s intention to represent series or ranges of cases so as to include boundary cases and maximally exhibit the rational interconnections among them all depends on the tolerance of an ineluctable ambiguity in the characteristic. Some of the boundary cases involve the infinitesimal, but some involve the infinitary. Scholars often say that while Aristotle abhorred the infinite and set up his conceptual schemata so as to exclude and circumvent it, Leibniz embraced it and chose conceptual schemata that could give it rational expression. This is true, and accounts for the way in which Leibniz devises and elaborates his characteristics in order to include infinitary as well as infinitesimalistic cases; but it has not been noticed that this use renders his characteristic essentially ambiguous. And he says as much.

He notes that the use of ambiguously finite / infinitesimal lines had been invoked by Guldin, Gregory of St. Vincent and Cavalieri, while the use of ambiguously finite / infinite lines was much less frequent, though not unknown: ‘For long ago people noticed the admirable properties of the asymptotes of the hyperbola, the conchoid, the cissoid, and many others, and the geometers knew that one could say in a certain manner that the asymptote of the hyperbola, or the tangent drawn from the center to that curve, is an infinite line equal to a finite rectangle… and in order to avoid trouble apropos the example we are using in order to try out this method, we will find in what follows that the *latus tranversum* of the parabola must be conceived as an infinite length."[9] Leibniz alludes to the fact that if we examine a hyperbola (or rather, one side of one of its branches) and the corresponding asymptote, the drawing must indicate both that the hyperbola continues ad infinitum, as does the asymptote, and that they will meet at the ideal point of infinity; moreover, a rule for calculating the area between the hyperbola and the asymptote (identified with the x-axis) can be given. The two lines may both be infinite, but their relation can be represented in terms of a finite (though ambiguous) notation—involving both letters and curves—and can play a determinate role in problems of quadrature. In the spring of 1673, Leibniz had traveled to London, where John Pell referred him to Nicolaus Mercator's *Logarithmotechnia*, in which Leibniz discovered Mercator's series. Taking his lead from the result of Gregory of St. Vincent, that the area under the hyperbola $y = 1 / (1 + x)$ from $x = 0$ to $x = x$ is what we now call $\ln (1 + x)$, Mercator represented the latter by the series $x / 1 - x^2 / 2 + x^3 / 3 - x^4 / 4 + \ldots$.

The more important example is that of the parabola; at stake are its relations to the other conic sections. Leibniz gives the following account of how to find a 'universal equation' that will unify and exhibit the relations among a series of cases. He offers as an illustration the conic sections, and what he writes is an implied criticism of Descartes' presentation of them in the *Geometry*, which does not sufficiently exhibit their harmony: 'The formation of a universal equation which must comprehend a number of particular cases will be found by setting up a *list of all the particular cases*. Now in order to make this list we must reduce everything to a line segment or magnitude, whose value is sought, and which must be determined by means of certain other line segments or magnitudes, added or subtracted; consequently there must be certain fixed points, or points taken as fixed, and others which move, whose possible different locations give us the catalogue of all the possible cases… having found this list, we must try to reduce all the possible cases to a general formula, by means of ambiguous signs, and of letters whose values are sometimes finite, sometimes infinitely large or small. I dare to claim that there is nothing so mixed up or ill-assorted that can't be reduced to harmony by this means.'[10]

He gives a diagram, with a bit of curved line representing an arbitrary conic section descending to the right from the point A, ABYE; a vertical axis AXDC descending straight down from A; and perpendicular to that axis at X another axis XY which meets the curve in Y; the line DE is drawn parallel to XY. Two given line segments *a* and *q* represent the parameters of the conic section. Leibniz asserts that the general equation for all the cases, where AX = x and XY = y, must then be,

$$2ax \mp (a/q)\,x^2 - y^2 = 0$$

When *a* and *q* are equal and $\mp$ is explicated as –, we have the circle of radius $a = q$; when *a* and *q* may be equal or unequal and $\mp$ is explicated as –, we have an ellipse where *a* is the *latus rectum* and *q* is the *latus transversum*; when $\mp$ is explicated as +, the conic section is the hyperbola.

However, in order to include further both the parabola and the straight line as cases of the conic section, Leibniz asserts, one must make use of infinite or infinitely small lines. 'Now supposing that the line *q*, or the *latus transversum* of the parabola be of infinite length, it is clear that the equation $2axq \mp ax^2 = qy^2$, will be equivalent to this one: $2axq = qy^2$ (which is that of the parabola) because the term ax^2 of the equation is infinitely small compared to the others $2axq$, et qy^2…'[11] And with respect to the straight line, he asserts, we must take both *a* and *q* as being infinitely small, that is, infinitesimal. 'Consequently, in the equation: $2ax \mp (a/q)\,x^2 = y^2$, the term 2ax will vanish as it is infinitely small compared to $(a/q)\,x^2$ et y^2, and that which remains will be + (a / q) $x^2 = y^2$ with the sign ╪ changed into +. Now the ratio of two infinitely small lines may be the same as that of two finite lines and even of two squares or of two rectangles; thus let the ratio a / q be equal to the ratio e^2 / d^2 and we will have (e^2 / d^2) $x^2 = y^2$ or (e / d) x = y whose locus is the straight line.'[12] Leibniz concludes that this equation, by exhibiting the conic sections as limit cases of one general equation, not only displays their mutual relations as a coherent system, but also explains many peculiar features of the special cases: why only the hyperbola has asymptotes, why the parabola and the straight line do not have a center while the others do, and so forth.

At the end of the essay, Leibniz notes that we must distinguish between ambiguity which is an equivocation, and ambiguity which is a 'univocation.' The ambiguity of the sign ╪ is an example of equivocation which must be eliminated each time we determine the general equation with respect to the special cases. But the ambiguity of the *letters* must be retained; it is the way the characteristic expresses the Principle of Continuity, for Leibniz believed that the infinitesimal, the finite, and the infinite are all subject to the same rational constraints. One rule will embrace them, but it must be written in an irreducibly ambiguous idiom. 'With respect to signs [for operations], the interpretation must free the formula from all equivocation. For we must consider the ambiguity that comes from letters as giving a 'univocation' or universality but that which comes from signs as producing a true equivocation, so that a formula that only contains ambiguous letters gives a truly general theorem… The first kind of interpretation is without difficulty, but the other is as subtle as it is important, for it gives us the means to create theorems and absolutely universal constructions, and to find general properties, and even definitions and subaltern kinds common to all sorts of things which seem at first to be very distant from each other… it throws considerable illumination on the harmony of things.[13] We should not think that Leibniz wrote this only in the first flush of his mathematical discoveries, and that the more sophisticated notations and more accurate problem-solving methods which he was on the threshold of discovering would dispel this enthusiasm for productive ambiguity. A look at two of his most celebrated investigations of transcendental curves by means of his new notation will prove my point.

3. The Principle of Perfection

Leibniz's definition of perfection is the greatest variety with the greatest order, a marriage of diversity and unity. He compares the harmonious diversity and unity among monads as knowers to different representations or drawings of a city from a multiplicity of different perspectives, and it is often acknowledged that this metaphor supports an extension to geographically distinct cultural groups of people who generate diverse accounts of the natural world, which might then profitably be shared. However, it is less widely recognized that this metaphor concerns not only knowledge of the contingent truths of nature but also moral and mathematical truths, necessary truths. As Frank Perkins argues at length in Chapter 2 of his *Leibniz and China: A Commerce of Light*, the human expression of necessary ideas is conditioned (both enhanced and limited) by cultural experience and embodiment, and in particular by the fact that we reason with other people with whom we share systems of signs, since for Leibniz all human thought requires signs. Mathematics, for example, is carried out within traditions that are defined by various modes of representation, in terms of which problems and methods are articulated.

After having set out his textual support for the claim that on Leibniz's account our monadic expressions of God's ideas and of the created world must mutually condition each other, Perkins sums up his conclusions thus: ‘We have seen... that in its dependence on signs, its dependence on an order of discovery, and its competition with the demands of embodied experience, our expression of [necessary] ideas is conditioned by our culturally limited expression of the universe. We can see now the complicated relationship between the human mind and God. The human mind is an image of God in that both hold ideas of possibles and that these ideas maintain set relationships among themselves in both. Nonetheless, the experience of reasoning is distinctively human, because we always express God’s mind in a particular embodied experience of the universe. The human experience of reason is embodied, temporal, and cultural, unlike reason in the mind of God.’[14] Innate ideas come into our apperception through conscious experience, and must be shaped by it.

With this view of human knowledge, marked by a sense of both the infinitude of what we try to know and the finitude of our resources for knowing, Leibniz could not have held that there is one correct ideal language. And Leibniz’s practice as a mathematician confirms this: his mathematical *Nachlass* is a composite of geometrical diagrams, algebraic equations taken singly or in two-dimensional arrays, tables, differential equations, mechanical schemata, and a plethora of experimental notations. Indeed, it was in virtue of his composite representation of problems of quadrature in number theoretic, algebraic and geometrical terms that Leibniz was able to formulate the infinitesimal calculus and the differential equations associated with it, as well as to initiate the systematic investigation of transcendental curves.[15] Leibniz was certainly fascinated by logic, and sought to improve and algebraize logical notation, but he regarded it as one formal language among many others, irreducibly many. Once we admit, with Leibniz, that expressive means that are adequate to the task of advancing and consolidating mathematical knowledge must include a variety of modes of representation, we can better appreciate his investigation of transcendental curves, and see why and how he went beyond Descartes.

4. Transcendental Curves: The Isochrone and the Tractrix

Leibniz's study of curves begins in the early 1670's when he is a Parisian for four short years. He takes up Cartesian analytic geometry (modified and extended by two generations of Dutch geometers including Schooten, Sluse, Hudde, and Huygens) and develops it into something much more comprehensive, analysis in the broad 18th century sense of that term. Launched by Leibniz, the Bernoullis, L'Hôpital and Euler, analysis becomes the study of algebraic and transcendental functions and the operations of differentiation and integration upon them, the solution of differential equations, and the investigation of infinite sequences and series. It also plays a major role in the development of post-Newtonian mechanics.

The intelligibility of geometrical objects is thrown into question for Leibniz in the particular form of (plane) transcendental curves: the term is in fact coined by Leibniz. These are curves that, unlike those studied by Descartes, are not algebraic, that is, they are not the solution to a polynomial equation of finite degree. They arise as isolated curiosities in antiquity (for example, the spiral and the cycloid), but only during the seventeenth century do they move into the center of a research program that can promise important results. Descartes wants to exclude them from geometry precisely because they are not tractable to his method, but Leibniz argues for their admission to mathematics on a variety of grounds, and over a long period of time. This claim, of course, requires some accompanying reflection on their conditions of intelligibility.

For Leibniz, the key to a curve's intelligibility is its hybrid nature, the way it allows us to explore numerical patterns and natural forms as well as geometrical patterns on the other; he was as keen a student of Wallis and Huygens as he was of Descartes. These patterns are variously explored by counting and by calculation, by observation and tracing, and by construction using the language of ratios and proportions. To think them all together in the way that interests Leibniz requires the new algebra as an *ars inveniendi*. The excellence of a characteristic for Leibniz consists in its ability to reveal structural similarities. What Leibniz discovers is that this 'thinking-together' of number patterns, natural forms, and figures, where his powerful and original insights into analogies pertaining to curves considered as hybrids can emerge, rebounds upon the algebra that allows the thinking-together and changes it. The addition of the new operators d and $\int$, the introduction of variables as exponents, changes in the meaning of the variables, and the entertaining of polynomials with an infinite number of terms are examples of this. Indeed, the names of certain canonical transcendental curves (log, sin, sinh, etc.) become part of the standard vocabulary of algebra.

This habit of mind is evident throughout Volume I of the VII series (*Mathematische Schriften*) of Leibniz's works in the Berlin Akademie-Verlag edition, devoted to the period 1672-1676.[16] As M. Parmentier admirably displays in his translation and edition *Naissance du calcul différentiel, 26 articles des* Acta eruditorum, the papers in the *Acta Eruditorum* taken together constitute a record of Leibniz's discovery and presentation of the infinitesimal calculus.[17] They can be read not just as the exposition of a new method, but as the investigation of a family of related problematic things, that is, algebraic and transcendental curves. In these pages, sequences of numbers alternate with geometrical diagrams accompanied by ratios and proportions, and with arrays of derivations carried out in Cartesian algebra augmented by new concepts and symbols.

For example, ‘De vera proportione circuli ad quadratrum circumscriptum in numeris rationalibus expressa,’[18] which treats the ancient problem of the squaring of the circle, moves through a consideration of the series $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9...$, to a number line designed to exhibit the finite limit of an infinite sum. Various features of infinite sums are set forth, and then the result is generalized from the case of the circle to that of the hyperbola, whose regularities are discussed in turn. The numerical meditation culminates in a diagram that illustrates the reduction: in a circle with an inscribed square, one vertex of the square is the point of intersection of two perpendicular asymptotes of one branch of a hyperbola whose point of inflection intersects the opposing vertex of the square. The diagram also illustrates the fact that the integral of the hyperbola is the logarithm. Integration takes us from the domain of algebraic functions to that of transcendental functions; this means both that the operation of integration extends its own domain of application (and so is more difficult to formalize than differentiation), and that it brings the algebraic and transcendental into rational relation.

During the 1690s, Leibniz investigates mathematics in relation to mechanics, deepening his command of the meaning and uses of differential equations, transcendental curves and infinite series. In this section I will discuss two of these curves, the isochrone and the tractrix, and in the next, the catenary. The isochrone is the line of descent along which a body will descend at a constant velocity. Leibniz publishes his result in the *Acta Eruditorum* in 1689 under the title, ‘Courbe isochrone, sur laquelle un corps pesant descend sans être accéléré; controverse avec l’abbé D. C.’[19] However, the real analysis of the problem is found in a manuscript published by Gerhardt in the *Mathematische Schriften* V (241-243), and accompanied by two diagrams: the first, reversed, is incorporated in the second.

<Figure 1 about here>

On the first page of this text, the diagram labled 119 [Figure 1] is read as infinitesimal. It begins, ‘The line of descent called the isochrone YYEF is sought, in which a heavy body descending on an incline approaches the plane of the horizon uniformly or isochronously, that is, so that the times are equal, in which the body traverses BE, EF, the perpendicular descents BR, RS being assumed equal. Let YY be the line sought, for which AXX is the straight line directrix, on which we erect perpendiculars; let us call *x* the abscissa AX, and let us call *y* the ordinate XY, and ${}_1X_2X$ or ${}_1Y_1D$ will be d*x* and let ${}_1D_2Y$ be called d*y*.’[20] The details of the analysis are interesting, as Leibniz works out a differential equation for the curve and proves by means of it what was in fact already know, that the curve is a quadrato-cubic paraboloid. However, what matters for my argument here is that we are asked to read the diagram as infinitesimalistic, since ${}_1X_2X$, ${}_1Y_1D$, and ${}_1D_2Y$ are identified as differentials.

<Figure 2 about here>

Immediately afterwards, in the section labeled ‘Problema, Lineam Descenscoriam isochronam invenire,’ exactly the same diagram is used, but reversed, incorporated into a larger diagram, and with some changes in the labeling. Here, by contrast, the diagram labeled 120 [Figure 2] is meant to be read as a finite configuration; but it intended to be the same diagram. Note how Leibniz begins: ‘Let the line BYYEF be a quadrato-cubic paraboloid, whose vertex is B and whose axis is BXXRS…’ There is no S in Figure 2; but the argument that follows makes sense if we suppose that ‘G’ ought to be ‘S’ as it is in Figure 1. Leibniz shows, using a purely geometrical argument using the idiom of proportions, that if the curve is the quadrato-cubic paraboloid, then it must be the isochrone. A heavy object falling from B along the line BYY, given its peculiar properties, must fall in an isochronous manner: ‘namely, the ratio between the time in which the heavy object runs down along line BYY from B to E, and the time in which it runs down from E to F, will be [the same as] the ratio of BR to RS; and then if BR and RS are equal, so also the intervals of time, in which it descends from B to E and from E to F, will be equal.'[21] What we find here is the same diagram employed in two different arguments that require it to be read in different ways; what a diagram means depends on its context of use. We might say that in the second use here, the diagram is iconic, because it resembles the situation it represents directly, but in the first use it is symbolic, because it cannot directly represent an infinitesimalistic situation. Yet the sameness of shape of the curve links the two employments, and holds them in rational relation.

We can find other situations in which the same diagram is read in two ways within the same argument. The tractrix is the path of an object dragged along a horizontal plane by a string of constant length when the end of the string not joined to the object moves along a straight line in the plane; you might think of someone walking down a sidewalk while trying to pull a recalcitrant small dog off the lawn by its leash. In fact, in German the tractrix is called the *Hundkurve*. The Parisian doctor Claude Perrault (who introduces the curve to Leibniz) uses as an example a pocket watch attached to a chain, being pulled across a table as its other end is drawn along a ruler. The key insight is that the string or chain is always tangent to the curve being traced out; the tractrix is also sometimes called the 'equitangential curve' because the length of a tangent from its point of contact with the curve to an asymptote of the curve is constant. The evolute of the tractrix is the catenary, which thus relates it to the quadrature of the hyperbola and logarithms.[22] So the tractrix is, as one might say, well-connected.

<Figure 3 about here>

Leibniz constructs this curve in an essay that tries out a general method of geometrical-mechanical construction, 'Supplementum geometriae dimensoriae seu generalissima omnium tetragonismorum effectio per motum: similiterque multiplex constructio lineae ex data tangentium conditione,' published in the *Acta Eruditorum* in September, 1693.[23] His diagram, like the re-casting of Kepler's Law of Areas in Proposition I, Book I, of Newton's *Principia*, represents a curve that is also an infinite-sided polygon, and a situation where a continuously acting force is re-conceptualized as a series of impulses that deflect the course of something moving in a trajectory. The diagram labeled 139 must thus be read in two ways, as a finite *and* as an infinitesimal configuration. [Figure 3] Here is the accompanying demonstration: 'We trace an arbitrarily small arc of a circle $_3$AF, with center $_3$B, whose radius is the string $_3$A$_3$B. We then pull on the string $_3$BF at F, directly, in other words along its own direction towards $_4$A, so that from position $_3$BF it moves to $_4$B$_4$A. Supposing that we have proceeded from the points $_1$B and $_2$B in the same fashion as from $_3$B, the trace will have described a polygon $_1$B$_2$B$_3$B and so forth, whose sides always fall on [semper incident in] the string. From this stage on, as the arc $_3$AF is indefinitely diminished and finally allowed to vanish—which is produced in the continuous tractional motion of our trace, where the lateral displacement of the string is continuous but always *unassignable* [*inassignabilis*]—it is clear that the polygon is transformed into a curve having the string as its tangent.'[24]

Up to the last sentence, we can read the diagram as the icon of a finite configuration; in the last sentence, where the diagram becomes truly dynamical in its meaning, we are required to read it as the symbol of an infinitesimalistic configuration, a symbol that nonetheless reliably exhibits the structure of the item represented. (A polynomial is also a symbol that reliably exhibits the structure of the item it represents.) After Leibniz invents the d*x* and ∫ notation, his extended algebra can no longer represent mathematical items in an ambiguous way that moves among the finite, infinitesimal, and infinitary; thus, he must employ diagrams to do this kind of bridging for him. In the foregoing argument, and in many others like it, we find Leibniz exploiting the productive ambiguity of diagrams that link the finite and the infinitesimal in order to link the geometrical and dynamical aspects of the problem.

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Notes

[1] A. E. Supplementa 1713, t. V, section 6; M.S. V, 382-387.

[2] "Réplique à l'abbé D.C. sous forme de letter à Bayle," Feb. 1687; P.S. III, 45.

[3] Some commentators have been puzzled by Leibniz's allegiance to the notion of ratio and proportion. Marc Parmentier, for example, writes, "nous devons nous rappeler que les mathématiques de l'époque n'ont pas encore laïcisé les antiques connotations que recouvre le mot ratio. A cette notion s'attache un archaïsme, auquel l'esprit de Leibniz par ailleurs si novateur, acquitte ici une sorte de tribute, en s'obstinant dans une position indéfendable. La ratio constitue à ses yeux une entité séparée, indépendante de la fraction qui l'exprime ou plus exactement, la mesure. En ce domaine l'algèbre n'a pas encore appliqué le rasoir d'Occam. La preuve en est que la ratio était encore le support de la relation d'analogie, equivalence de deux

rapports, toute différente de la simple égalité des produits des extremes et des moyens dans les fractions." (G. W. Leibniz, *La naissance du calcul differential*, ed. M. Parmentier, Paris: Vrin, 1989, 42)

[4] "Ich kann dieser Vermutung hier nicht nachgehen und möchte mich mit der Feststellung begnügen, dass Leibniz zwar ein dem Logizismus entsprechendes philosophisches Programm vertreten hat, dass er aber durch seine Mathematik selbst sich weit von diesem Programm entfernt hat…" H. Breger, "Weyl, Leibniz, und das Kontinuum," *Studia Leibnitiana* Supplementa 26, 316-330.

[5] …donc pour concevoir que la ligne DE est la touchante, il faut seulement d'imaginer que la ligne B(B) ou la distance des deux points ou elle coupe est infiniment petite: et cela suffit pour trouver les tangentes. (C, 105)

[6] "le point C qui est *ambulatoire* peut tomber dans le point B." (C106)

[7] "afin que l'equation ne contradise pas l'egalité entre AC et AB." (C, 106)

[8] "Puisque on peut placer 3C, non seulement directement sous B, pour faire AC = AB et BC egale à rien, mais on le peut aussi placer en deça entre A, et B en (3C) ou au dela de B, en ((3C)) pour verifier par l'une des positions l'Equation AC = + AB – BC et par l'autre l'Equation AC = + AB + BC. pourveu que la ligne (3C)B ou ((3C))B soit conceüe infiniment petite. Voilà comment cette observation peut servir à la methode de l'universalité pour appliquer une formule generale à un cas particulier." (C 106)

[9] "Car il y a longtemps qu'on a observé les admirable proprietez des lignes Asymptotes de l'Hyperbole, de la Conchoeide, de la Cissoeide, et de plusieurs autres, et les Geometres n'ignorent pas qu'on peut dire en quelque façon que l'Asymptote de l'Hyperbole, ou la touchante menée du centre à la courbe est une ligne infinie egale à un rectangle fini... Et pour ne

pas prevenir mal à propos l'exemple dont nous nous servirons pour donner un essay de cette methode, on trouvera dans la suite, que *latus transversum* de la parabole doit estre conceu d'une longueur infinie." (C 106-107)

[10] La formation d'une Equation Universelle qui doit comprendre quantité de cas particuliers se trouvera en dressant une *liste de tous les cas particuliers*. Or pour faire cette *liste* il faut reduire tout à une ligne, ou grandeur, dont la valeur est requise, et qui se doit determiner par le moyen de quelques autres lignes ou grandeurs adjoustées ou soubstraites, par consequent il faut qu'il y ait certains *points fixes*, ou pris pour fixes,… et d'autres *ambulatoires*, dont les endroits possibles differents nous donnent le catalogue de tous les cas possibles… Ayant trouvé cette liste, il faut songer à reduire à une formule generale tous les cas possibles, par le moyen de signs ambigus, et des lettres dont la valeur est tantost ordinaire, tantost infiniment grande ou petite. J'ose dire qu'il n'y a rien de si brouillé, et different qu'on ne puisse reduire en harmonie par ce moyen… (C 114-115)

[11] Or posons que la ligne, q, ou le latus transversum de la Parabole soit d'une longueur infinite, il est manifeste, que l'Equation $2axq$ ╪ $ax^2 = qy^2$, sera equivalente à celle cy: $2axq = qy^2$ (qui est celle de la Parabole) parce que le terme de l'Equation ax^2, est infiniment petit, à l'egard des autres $2axq$, et qy^2… (C 116)

[12] Par consequent dans l'Equation: $2ax$ ╪ $(a / q)\, x^2 = y^2$, le terme $2ax$ evanouira comme infiniment petit, à l'egard de $(a / q)\, x^2$ et y^2, et ce qui restera sera $+ (a / q)\, x^2 = y^2$ le signe ╪ estant change en + or la raison de deux lignes infiniment petites peut estre la mesme avec celle de deux lignes ordinaires et mesme de deux quarrez ou rectangles soit donc la raison a / q egale à la raison e^2 / d^2 et nous aurons $(e^2 / d^2)\, x^2 = y^2$ ou $(e / d)\, x = y$ dont le lieu tombe dans une droite. (C 116)

[13] A l’egard des signes, l’interpretation doit delivrer la formule de toute l’equivocation. Car il faut considerer que l’ambiguité qui vient des lettres donne une Univocation ou Universalite mais celle qui vient des signes produit une veritable equivocation de sorte qu’une formule qui n’a que des lettres ambigues, donne un theoreme veritablement general… La première sorte d’interpretation est sans aucune façon ni difficulté, mais l’autre est aussy subtile qu’importante, car elle nous donne le moyen de faire des theorems, et des constructions absolument universelles, et de trouver des proprietez generales, et mesme des definitions ou genres subalterns communs à toutes sortes de choses qui semble bien éloignées l’une de l’autres… celle-cy donne des lumieres considerables pour l’harmonie des choses. (C 119)

[14] Franklin Perkins, *Leibniz and China, A Commerce of Light*, Cambridge: Cambridge University Press, 2004, pp. 96-97.

[15] Emily Grosholz, “Was Leibniz a Mathematical Revolutionary?” in *Revolutions in Mathematics*, D. Gillies, ed., Oxford: Clarendon P02, Ch. 37.

[16] *G. W. Leibniz's Mathematische Schriften, Geometrie -- Zahlentheorie -- Algebra 1672-1676* (Series VII, Vol. 1), E. Knobloch and W. Contro, eds., Berlin: Akademie Verlag, 1990.

[17] M. Parmentier, *La naissance du calcul différentiel, 26 articles des Acta eruditorum*, Paris: J. Vrin, 1995.

[18] AE Feb. 1682, [19] *Acta Eruditorum* April 1689, MS V 234-237.

[20] “Quaeritur Linea descensoria isochrona YYEF (fig. 119), in qua grave inclinate descendens isochrone seu uniformiter plano horizontali appropinquet, ita nempe ut aequalibus temporibus, quibus percurrantur arcus BE, EF, aequales sint descensus BR, RS, in perpendiculari sumti. Sit linea quaesita YY, cujus recta Directrix, in qua ascensus perpendiculares metiemur, sit AXX; abscissa AX vocetur y, et ${}_1X_2X$ seu ${}_1Y_1D$ erit d*x* et ${}_1D_2Y$ vocetur d*y*. ” MS V, 241.

[21] "Nempe tempus quo grave ex B in linea BYY decurret ad E, erit ad tempus quo ex E decurret ad F, ut BR ad RS, ac proinde si BR et RS sint aequales, etiam temporis intervalla, quibus ex B descenditur in E et E in F, erunt aequalia." MS V, 242.

[22] The evolute of a given curve is the locus of centres of curvature of that curve. It is also the envelope of normals to the curve; the normal to a curve is the line perpendicular to its tangent, and the envelope is a curve or surface that touches every member of a family of lines or curves (in this case, the family of normals).

[23] MS V, 294-301.

[24] "Quod et sic demonstratur: Centro ${}_{3}B$ et filo ${}_{3}A_{3}B$ tanquam radio describatur arcus circuli utcunque parvus ${}_{3}AF$, inde filum ${}_{3}BF$, apprehensum in F, directe seu per sua propria vestigia trahatur usque ad ${}_{4}A$, ita ut ex ${}_{3}BF$ transferatur in ${}_{4}B_{4}A$; itaque si ponatur similiter fuisse processum ad puncta ${}_{1}B$, ${}_{2}B$, ut ad punctum ${}_{3}B$, utique punctum B descripsisset polygonum ${}_{1}B_{2}B_{3}B$ etc. cujus latera semper incident in filum, unde imminuto indefinite arcu, qualis erat ${}_{3}AF$, ac tandem evanescente, quod fit in motu tractionis continuae, qualis est nostrae descriptionis, ubi continua, sed semper *inassignabilis* fit circumactio fili, manifestum est, polygonum abire in curvam, cujus tangens est filum." MS V, 296.