Leibniz on Mathematics and Representation:

Knowledge through the Integration of Irreducible Diversity

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When Louis Couturat and Bertrand Russell enlisted Leibniz as the champion of logicism, in the search for a single perfect idiom for the truths of mathematics and science, they dismissed at the same time much of his writing on theology and metaphysics. This was of course a brutal triage that generations of scholars throughout the twentieth century have critically examined and rejected, but we have still not properly assessed how it distorts our understanding of Leibniz's account of mathematics and science. In this essay, I argue that Leibniz believed that mathematics is best investigated by means of a variety of modes of representation, often stemming from a variety of traditions of research, like our investigations of the natural world and of the moral law. I connect this belief with two case studies from Leibniz's mathematical research: his development of the infinitesimal calculus, and his investigations of transcendental curves, in particular the catenary, *la chainette*. Then I set his belief in the context of certain contemporary discussions about the best account of mathematical and scientific knowledge, where anti-logicist sentiment is evident and well-motivated.

Leibniz's definition of perfection is the greatest variety with the greatest order, a marriage of diversity and unity. He compares the harmonious diversity and unity among monads as knowers to different representations or drawings of a city from a multiplicity of different perspectives, and it is often acknowledged that this metaphor supports an extension to geographically distinct cultural groups of people who generate diverse accounts of the natural world, which might then profitably be shared. However, it is less widely recognized that this metaphor concerns not only knowledge of the contingent truths of nature but also moral and mathematical truths, necessary truths. As Frank Perkins argues at length in Chapter 2 of his *Leibniz and China: A Commerce of Light*, the human expression of necessary ideas is conditioned (both enhanced and limited) by cultural experience and embodiment, and in particular by the fact that we reason with other people with whom we share systems of signs, since for Leibniz all human thought requires signs. Mathematics, for example, is carried out within traditions that are defined by various modes of representation, in terms of which problems and methods are articulated.

After having set out his textual support for the claim that on Leibniz's account our monadic expressions of God's ideas and of the created world must mutually condition each other, Perkins sums up his conclusions thus: "We have seen... that in its dependence on signs, its dependence on an order of discovery, and its competition with the demands of embodied experience, our expression of [necessary] ideas is conditioned by our culturally limited expression of the universe. We can see now the complicated relationship between the human mind and God. The human mind is an image of God in that both hold ideas of possibles and that these ideas maintain set relationships among themselves in both. Nonetheless, the experience of reasoning is distinctively human, because we always express God's mind in a particular embodied experience of the universe. The human experience of reason is embodied, temporal, and cultural, unlike reason in the mind of God."[1] Innate ideas come into our apperception through conscious experience, and must be shaped by it.

With this view of human knowledge, marked by a sense of both the infinitude of what we try to know and the finitude of our resources for knowing, Leibniz could not have held that there is one correct ideal language. And Leibniz's practice as a mathematician confirms this: his mathematical *Nachlass* is a composite of geometrical diagrams, algebraic equations taken singly or in two-dimensional arrays, tables, differential equations, mechanical schemata, and a plethora of experimental notations. Indeed, it was in virtue of his composite representation of problems of quadrature in number theoretic, algebraic and geometrical terms that Leibniz was able to formulate the infinitesimal calculus and the differential equations associated with it, as well as to initiate the systematic investigation of transcendental curves.[2] Leibniz was certainly fascinated by logic, and sought to improve and algebraize logical notation, but he regarded it as one formal language among many others, irreducibly many.

Once we admit, with Leibniz, that expressive means that are adequate to the task of advancing and consolidating mathematical knowledge must include a variety of modes of representation, we can take up with renewed interest two related philosophical issues. First, we can show more precisely how and why the combination of distinct mathematical disciplines, each with its own traditions of representation, can be fruitful; this is what I do in sections 1 and 2, where I examine some aspects of Leibniz's development of the infinitesimal calculus. Second, we can see more clearly how mathematicians manage to refer as well as to analyze, by combining different modes of representation or exploiting the structural ambiguity of both icons and symbols. Sections 3 and 4 are devoted to this issue, apropos Leibniz's investigation of the catenary. The relation between these issues is simple: the advance of knowledge depends on our ability to indicate what we are talking about, as well as discovering new things to say about it.

1. Productive Combination and Constructive Ambiguity

Leibniz is often praised for his prescient appreciation of the important role that formal language plays in mathematical and scientific discovery. His conviction about the usefulness—indeed the indispensability—of characteristics to an a r s i n v e n i e n d i certainly stemmed from the great success of his infinitesimal calculus, which expanded the characteristic of Descartes= algebra to include the symbol for differentiation (d*x*) and for integration (I*x*dy) as well as notation for infinite sums, sequences, and series. The admiration bestowed on Leibniz at the turn of the last century by Couturat and Russell has however obscured two important features of Leibniz=s use of characteristics. Russell was committed to a program of logicism, which sought a unified language for logic and procedures for reducing all of mathematics to logic, via a reduction first of arithmetic to logic and then of geometry to arithmetic. Because Russell read Leibniz as a logicist, he missed Leibniz=s emphasis on the multiplicity of formal languages.[3] For that matter, he also seems to have missed Frege's emphasis on the multiplicity of the B e g r i f f s c h r i f t.[4] Different formal languages reveal different aspects of a domain of things and problems, and lend themselves better to certain domains. Russell's narrowly focussed vision remained fixed on a linear symbolic characteristic, and neglected the conceptual possibilities offered by various two and three dimensional representations, as well as the more iconic representations of geometry, topology (nascent in Leibniz's a n a l y s i s s i t u s), and mechanics.

Given his emphasis on justification at the expense of discovery, Russell also missed Leibniz=s insight that writing, the use of characteristics to express thought and analyze the conditions of intelligibility of things, allows us to say more than we know we are saying: the best characteristics have a kind of generative power, especially when they are used in tandem. This is the positive converse to the negative results of Gödel's incompleteness theorems. A good characteristic advances knowledge not only by choosing and exhibiting especially deep fundamental concepts, but also by allowing us to explore the analogies among disparate things, a practice which in the formal sciences tends to generate new intelligible things, things I have called "hybrids" in some of my writings.[5] Moreover, characteristics add themselves to the furniture of the universe, representing themselves iconically as new intelligible objects as well as representing other mathematical things symbolically. The investigation of conditions of intelligibility not only discovers order but also induces order: we add to the non-totalizable infinity of intelligible things as we search for conditions of intelligibility using a spectrum of characteristics. Thus, although Russell reads Leibniz as a formalist who tries to reduce the mathematical concrete to the abstract-logical, and Benson Mates reads him as a nominalist who tries to reduce the mathematical abstract to the concrete-physical, he is in fact neither. His logical writings, his epistemological writings, and his mathematical practice offer instead a thinker who was very sensitive to the rational management of disparate modes of representation, and whose ontology is multi-leveled.

Leibniz's study of curves begins in the early 1670's when he is a Parisian for four short years. He takes up Cartesian analytic geometry (modified and extended by two generations of Dutch geometers including Schooten, Sluse, Hudde, and Huygens) and develops it into something much more comprehensive, analysis in the broad 18th century sense of that term. Launched by Leibniz, the Bernoullis, L'Hôpital and Euler, analysis becomes the study of algebraic and transcendental functions and the operations of differentiation and integration upon them, the solution of differential equations, and the investigation of infinite sequences and series. It also plays a major role in the development of post-Newtonian mechanics.

The intelligibility of geometrical objects is thrown into question for Leibniz in the particular form of (plane) transcendental curves: the term is in fact coined by Leibniz. These are curves that, unlike those studied by Descartes, are not algebraic, that is, they are not the solution to a polynomial equation of finite degree. They arise as isolated curiosities in antiquity (for example, the spiral and the cycloid), but only during the seventeenth century do they move into the center of a research program that can promise important results. Descartes wants to exclude them from geometry precisely because they are not tractable to his method, but Leibniz argues for their admission to mathematics on a variety of grounds, and over a long period of time. This claim, of course, requires some accompanying reflection on their conditions of intelligibility.

For Leibniz, the key to a curve's intelligibility is its hybrid nature, the way it allows us to explore numerical patterns and natural forms as well as geometrical patterns on the other; he was as keen a student of Wallis and Huygens as he was of Descartes. These patterns are variously explored by counting and by calculation, by observation and tracing, and by construction using the language of ratios and proportions. To think them all together in the way that interests Leibniz requires the new algebra as an a r s i n v e n i e n d i. The excellence of a characteristic for Leibniz consists in its ability to reveal structural similarities. What Leibniz discovers is that this "thinking-together" of number patterns, natural forms, and figures, where his powerful and original insights into analogies pertaining to curves considered as hybrids can emerge, rebounds upon the algebra that allows the thinking-together and changes it. The addition of the new operators d and $\int$, the introduction of variables as exponents, changes in the meaning of the variables, and the entertaining of polynomials with an infinite number of terms are examples of this. Indeed, the names of certain canonical transcendental curves (log, sin, sinh, etc.) become part of the standard vocabulary of algebra.

This habit of mind is evident throughout Volume I of the VII series (Mathematische Schriften) of Leibniz's works in the Berlin Akademie-Verlag edition, devoted to the period 1672-1676.[6] As M. Parmentier admirably displays in his translation and edition *Naissance du calcul différentiel, 26 articles des* Acta eruditorum, the papers in the *Acta Eruditorum* taken together constitute a record of Leibniz's discovery and presentation of the infinitesimal calculus.[7] They can be read not just as the exposition of a new method, but as the investigation of a family of related problematic things, that is, algebraic and transcendental curves. In these pages, sequences of numbers alternate with geometrical diagrams accompanied by ratios and proportions, and with arrays of derivations carried out in Cartesian algebra augmented by new concepts and symbols. For example, "De vera proportione circuli ad quadratrum circumscriptum in numeris rationalibus expressa,"[8] which treats the ancient problem of the squaring of the circle, moves through a consideration of the series $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9...$, to a number line designed to exhibit the finite limit of an infinite sum. Various features of infinite sums are set forth, and then the result is generalized from the case of the circle to that of the hyperbola, whose regularities are discussed in turn. The numerical meditation culminates in a diagram that illustrates the reduction: in a circle with an inscribed square, one vertex of the square is the point of intersection of two perpendicular asymptotes of one branch of a hyperbola whose point of inflection intersects the opposing vertex of the square. The diagram also illustrates the fact that the integral of the hyperbola is the logarithm. Integration takes us from the domain of algebraic functions to that of transcendental functions; this means both that the operation of integration extends its own domain of application (and so is more difficult to formalize than differentiation), and that it brings the algebraic and transcendental into rational relation.

2. Leibniz's Discovery of the Infinitesimal Calculus

The exploration of curves as hybrids is just what Leibniz describes in his own retrospective account of the intellectual genesis of the calculus, "Historio et origo calculi differentialis" (1714)[9] and the April 1703 letter to Bernoulli,[10] as well as the much earlier "De geometria recondita et analysi indivisibilium atque infinitorum," (1686)[11] where however part of the story is told in much the same way. In the "Historia et origo," Leibniz recounts—in the third person—that his initial mathematical discoveries were in arithmetic: "He took a keen delight in the properties and combinations of numbers; indeed, in 1666 he published an essay, "De arte combinatoria." Leibniz's abstractive, generalizing cast of mind took him immediately from the consideration of particular series of numbers to the study of their general relational properties. For example if, from a given series, one forms a second series of the differences holding between the original terms, what relations will hold between the given and the resultant series? The same question can be raised for the sum series, and then the operations of forming the difference series and the sum series are discovered to be inverse operations. Leibniz's first mathematical discovery of note during his sojourn in Paris is that the sum of consecutive terms in a series of differences is equal to the difference between the two extreme terms of the original series. This insight first attracts Huygens' interest in the young Leibniz; and his friendship proves essential to Leibniz's development as a mathematician.

Huygens' tutorial friendship with Leibniz introduces the young philosopher to new mathematical realms. Huygens advises Leibniz to study *inter alia* the work of Pascal, and in the latter's "Traité des sinus du quart de cercle" where a surface of rotation generated by a circle is geometrically shown to be proportional to an area created by "applying" the normals to the circle (here, the radii) to the y-axis, that is, by setting them perpendicular to the y-axis in order.[12] In the "De geometria recondita," Leibniz recalls, "While I was still a novice in these matters, it happened that, in the simple consideration of an argument on the measurement of a spherical surface, I suddenly saw a great light. I observed that, in full generality, the figure generated by the normals to a curve applied to the y-axis is proportional to the surface of rotation generated by rotating that curve around the y-axis. Transported by joy at this first theorem (ignorant though I was that no one else had ever discovered anything like it), I soon postulated for all curves a triangle which I called the characteristic triangle, whose sides were indivisibles (or, to speak more precisely, infinitely small), that is to say, differential quantities; and I deduced from this straightway quite a few theorems that subsequently I discovered piecemeal in the works of Gregory and Barrow. But I did all this without making use of an algebraic calculus."[13] The "application" of the normals to the given curve onto the y-axis generates a new curve, which is by that very process as we would say integrated; this quantity is then affirmed to be proportional to the surface of rotation generated by the original curve, another integral. This is an elegant general method, but of course at that point Leibniz had no way of characterizing many of the new curves generated by the process of the application of normals, nor did he know how to find the areas under such curves. His reasoning, like that of Pascal, hinged on the similarity between a finite triangle and a characteristic triangle associated with the curve; he had picked up the notion of Cavalierian indivisibles, but as yet had no explicit algebraic language for the situation.

For Leibniz, finite combinatorial arithmetic points beyond itself to the study of infinite series, and geometry to the study of curves outside the classical canon, precisely because of his intellectual habit of generalization. But he was well aware that he cannot consolidate his own results. At this crucial juncture, Huygens sent him back to the library: “He told me to read the works of Descartes and Slusius, who showed how to form equations for loci.”[14] There he discovered the powerful and expressive language of Cartesian algebra in Schooten's two-volume edition of the *Geometry*. Algebra furnished two essential devices for the development of Leibniz's thoughts: it allowed him to express perspicuously the rule for an infinite series, and to associate curves with polynomial equations. The idiom of Cartesian algebra confers a new kind of unity on patterns of numbers, and brings out a new dimension of the unity of a curve, the way its shape constrains various geometrical magnitudes associated with the curve (like its normals, tangents, areas inscribed beneath it, abscissas, and so forth) and makes them vary in tandem. It also allows the unity of patterns of numbers, and the unity of a curve, to be thought together in a new way.

One of the first fruits of Leibniz's use of the new algebra was his solution to the old problem of the squaring of the circle. Of course, this is not a solution in the strict sense—that being impossible—but it does show how to express the area of a circle in terms of a simple numerical pattern: $\pi/4 = 1 - 1/3 + 5 - 1/7 + 1/9 - \ldots$. As set forth in “De vera proportione circuli ad quadratum circumscriptum in numeris rationalibus expressa,” the proof uses his geometric insight gained through the study of Pascal to obtain the circular area by means of the quadrature of $y = xx / 1 + xx$, making use of Mercator's expression of $1 / 1 + t$ as $1 - t + t^2 - t^3 + t^4 - \ldots$. But the very question, why the quadrature of curves should be bivalently thinkable in terms of both geometrical construction and series of numbers, cannot be answered by the restricted idiom of

Cartesian algebra, which brings the question into view but then leaves it suspended. As Leibniz writes in "De geometria recondita," "When I began to use [the algebraic calculus], I didn't waste any time in discovering my arithmetic quadrature, and many other things. But, who knows why, the algebraic calculus in these matters didn't entirely bring me satisfaction, constrained as I was to pass by the detour of a figure for establishing many results that I would have liked to obtain by analysis; up to the day when I finally discovered the true complement of algebra for transcendent things, my infinitesimal calculus, or as I also call it, my differential calculus, integral calculus, and—rather judiciously, I'd say—the analysis of indivisible and infinites."[15] The application of Cartesian algebra to "transcendent things" changes algebra itself to produce the "infinitesimal calculus," whose immediate consequence is to generalize and simplify, as well as to offer a deeper-lying explanation of why algebra brings together the study of series of numbers and that of curves so effectively.

In *Historia et origo calculi differentialis*, Leibniz's account of his first formulation of the 'd' notation ('d' for differential—Leibniz thinks of dx as a line segment smaller than any finite line segment) refers to the arithmetic problems that first sparked his interest in mathematics.[16] He describes Pascal's triangle and his own harmonic triangle, and points out that these arrays represent infinitely extended series of (natural or rational) numbers set alongside their sum or difference series. Using his 'd' notation and Cartesian algebra to represent the general term of a series, Leibniz saw that he could formulate the relations among these series as rules. Thus, if the general term of the initial series is x^2, its difference series is $\mathrm{d}x^2 = x^2 + 2x + 1$. Finding an expression for the difference series of an initial series with the form of a polynomial is straightforward; finding an expression for the sum series is not; a general method can be articulated "only if the value of the general term can be expressed by means of a variable x so that the variable does not enter into a denominator or an exponent." Again, if the general terms of the initial series is x^3, its sum series is $x^3/3 - x^2/2 + x/6$. In general, the task of finding sum series leads directly to the question of which series converge and which do not. Leibniz makes it clear that the formation of difference series and that of sum series are inverse operations.[17]

The hybrid that Leibniz uses to make the leap from the combinatorial, arithmetical context to the geometrical study of curves is that of the curve as infinite-sided polygon. The initial series, x, or $y = x^2$, is no longer a discrete series of integers or rational numbers that might be thought of as labels for the vertices of the polygon; rather, it stands for "all the abscissas" or "all the ordinates" of the curve; and $\mathrm{d}x$ or $\mathrm{d}y$ then stands for differences that are infinitesimal, smaller than any finite magnitude. Leibniz's faith in the intelligibility of things is well repaid here, for the analogy between the finitary and the infinitary, carefully pursued, holds remarkably well. Formulae for the differential calculus are found in a straightforward manner; formulae for the integral calculus, as in the finitary case, prove much more difficult to discover as integration typically leads from known to unknown curves. Once again, the sum of consecutive terms of a difference series is equal to the difference of the two extreme terms of the original series; and the operations of differentiation and integration are mutually inverse. New curves, in particular transcendental curves, may be defined and investigated by means of differential equations.

The array of numbers that constitute Pascal's triangle and the problem of squaring the circle are ancient topics for mathematical meditation, arising independently in a number of different cultures. Leibniz is able to think these patterns together, by using the new algebra and extending it by notions that allow the mathematician to pass via the infinitesimal and infinitary to return to the finite, as the expression $\int \mathrm{d}x = x$ shows so concisely. This inspired detour allows one to transfer insights about finitary, combinatorial items to the continuous items of geometry: the hybrid curve q u a infinite sided polygon—at once geometrical, arithmetic, and algebraic—by holding together different domains also brings the infinitesimal, finite, and infinite into rational relation. The intelligible unity of a transcendental function, for example, can be represented by the unity of its peculiar shape (which constrains various magnitudes associated with the curve in characteristic ways), and by the patterns in the numbers associated with the curve; moreover, adumbrated by either shape or number patterns, it can be represented by algebraic form, which holds the geometrical and the numerical unities together as distinct concrete expressions. And that algebraic form exists both as the differential equation that defines the curve (indeed, a whole family of curves) and as the equation that constitutes the solution of the differential equation; the former expresses the defining conditions for the curve and the latter the structural features of the curve itself.

In other words, it is a mistake to suppose that, as we would say now, once a function has been given analytically that its equation is the sole condition of its intelligibility. The intelligibility of algebra rests on other, more concrete modes of representation, which depend for their meaning on the intelligible autonomy of geometry and that of arithmetic, although insofar as algebra applies both to Euclidean geometric patterns and to numerical patterns in the real numbers (and to other groups, rings and fields, as mathematicians were to discover), it must also be allowed its own relative autonomy. The infinitesimal calculus, for example, is a development of Cartesian algebra rather than of arithmetic or geometry; it is called for because of what Cartesian algebra makes visible and problematic, and the limitations of that algebra. This does not mean, of course, that geometry and arithmetic are not implicated in this development.

Different domains are constituted as problems that arise with respect to certain problematic objects: arithmetic arises as problems related to number and geometry as problems related to figure. But it is also characteristic of mathematical domains to allow the articulation of problems that cannot be solved within that domain, whose solutions both extend the boundaries of the domain and bring that domain into partial structural analogy with another domain or domains. The conditions of intelligibility in mathematics are closely tied to conditions of solubility of problems; the examination of the latter gives clues to the search for the former. But conditions of intelligibility also include unified and unifying forms like algebra that allow distinct domains to be brought into explicit rational relation. The way that algebra allows problems to be solved is to offer abstractive forms -- with their own tractable problematicity -- that make solution strategies from the distinct domains available concurrently as well as strategies suggested by the new abstractive forms. The conditions of intelligibility for problematic hybrids are similarly complex, and pertain to the related domains as well as to the more abstract, unified and unifying forms. The reflective search for conditions of intelligibility is thus accompanied by the historical study of how problems arise and are solved.

3. Pragmatic Considerations in the Growth of Mathematical Knowledge

Robin Hendry is one of the current generation of philosophers of science who criticize mid-twentieth-century Anglophone philosophy of science by calling for the introduction of pragmatical as well as semantical and syntactical concerns into our account of scientific rationality. This means that we must look at how representations in science and mathematics are used in a particular historical and theoretical context. In mathematics, as we have seen, the context is characterized first and foremost by a collection of solved and unsolved problems, a tradition of modes of representation, methods (including calculation and construction procedures) for addressing those problems, and standards for what may be a satisfactory solution to a problem. Summarizing his arguments why the notion of 'isomorphism' should not be given a leading role in the semantical account of truth as a relation among a theory and its models, Hendry writes, "Firstly, representation cannot be identified with isomorphism, because there are just too many relation-instances of isomorphism. Secondly, a particular relation-instance of isomorphism is a case of representation only in the context of a scheme of use that fixes what is to be related to what, and how. Thirdly, in reacting to the received [syntactic] view's linguistic orientation, the semantic view goes too far in neglecting language, because language is a crucial part of the context that makes it possible to use mathematics to represent. Natural languages afford us abilities to refer, and equations borrow these abilities. We cannot fully understand particular cases of representation in the absence of a 'natural history' of the traditions of representation of which they are a part.'[18]

When we try to speak truly about things, we ask language (broadly construed) to perform at least two different roles: to indicate what we are talking about, and to analyze it by discovering its conditions of intelligibility, its rational requisites, its reasons for being what it is.[19] Hendry is arguing that the semantical approach fails to account for how language fulfills both these functions in successful science and mathematics. The traditional schema for a proposition used by logicians is 'S is P,' and in general the subject term pertains to the first role, and the predicate term to the second. These two roles may be filled by two different modes of representation used in tandem, or one mode of representation used ambiguously; I have elsewhere exhibited mathematical episodes where this happens, in problem-solving by Descartes, and Galileo.[20] In such cases, natural language is also needed to explain the relation between the two modes, or the ambiguity of the one mode (by explaining the relation between its two uses).

The important asymmetry between S and P in the assertion of a proposition has been covered over by 20th century logic. I think this was because it could not admit the *variety* of kinds of things treated in mathematics, but sought rather to homogenize its subject matter: if mathematics is only about sets (for example) then there is no reason to distinguish between S and P, since both are merely sets. The problem then becomes, as Benaceraff admits in his often-cited essay "Mathematical Truth," that the dominant mode of representation has lost its power to refer.[21] If as philosophers and logicians, we pay attention to the asymmetry between S and P in a proposition, we see that language used for S typically plays the role of referring whereas language used for P typically plays the role of analyzing. The assertion of a proposition juxtaposes them.

Thus, the combination of modes of representation that Hendry discusses in the study of molecules, and that I discuss in the study of curves, points towards a deeper epistemological issue, if in the assertion of any truth we must use representations in two different ways. Iconic representations that picture help us refer, as do proper names, pronouns, and indexicals; and analysis is often carried out symbolically, since conditions of intelligibility are thought to hold of certain kinds of things universally. But a representation doesn't wear its function on its face; careful reading of the text in which it occurs is required, in order to see how it should be understood. Pictures sometimes function symbolically, and sometimes they combine an iconic function with a symbolic, analytic one. Sometimes the thing under investigation can only be given or encountered indirectly, and in consequence of an analysis of its conditions of intelligibility; then its indication by language comes last and may be highly symbolic. In mathematics, this is often the case with infinitary objects, like function spaces or groups of automorphisms of infinite sets or spaces.

The Russellian view that true knowledge ought to be expressed in a single, preferred, univocal idiom, and that subject and predicate terms have the same status, has made the multi-vocality and ambiguity that I have just claimed is required of veridical language almost impossible for certain philosophers to see. This is especially striking in philosophy of mathematics, where during the past century some of the most influential philosophers thought (erroneously) that they had invented such a language. As mentioned above, the compendium of essays, *The Philosophy of Mathematics* edited by W. D. Hart, designed to define "the state of the art" in the mid-1990s, begins with Paul Benacerraf's essay 'Mathematical Truth.' Benacerraf argues that if we think of the truth conditions of mathematical claims as their formal derivability from specified sets of axioms, then we can't explain how and why we know what we are talking about; that is, we can't explain the relation between theorem-hood and truth. Further, and this is the other horn of the dilemma, the only epistemology that seems to Benacerraf to explain successful referring is causal—which may work for medium-sized physical objects but not for the things of mathematics. Benacerraf's way of describing the situation makes it appear as if we are limited to a single formal language for mathematics, and a physical world where spatio-temporal things bump into us and make knowledge of them possible. All the other essays in the book respond to this dilemma without questioning its basic assumptions. I claim, however, that mathematics employs a number of formal modes of representation, which with the help of natural language may be used in tandem or used ambiguously to carry out different linguistic functions. If we were limited to only one, symbolic, axiomatized language for the expression of mathematical truth, we could not do it; but in fact mathematicians are not so limited. They are able to solve problems successfully because they can tether their polyvalent discourse to mathematical things in many ways; how this happens provides important clues for going beyond “naturalized epistemology” in order to find a theory of knowledge that will work properly for mathematics.

In *The Dappled World: A Study of the Boundaries of Science*, Nancy Cartwright’s meditation on why we must reflect more deeply on the relationship between the abstract and the concrete bears directly on the point I have just been developing.[22] She uses her argument to talk about episodes in modern physics, yet clearly what she says has a more general import for epistemology. Chapter 2 of her book recalls Lessing’s distinction between intuitive cognition, where we attend to our ideas of things (I would say, to our awareness of things) and symbolic cognition, where we attend to signs we have substituted for them. She writes, “All universal and general knowledge, all systematic knowledge, requires symbolic cognition; yet only intuitive cognition can guarantee truth and certainty. For words are arbitrary, you cannot see through them to the idea behind them,” and quotes Lessing: “Intuitive knowledge is clear in itself; the symbolic borrows its clarity from the intuitive… In order to give a general symbolic conclusion all the clarity of which it is capable, that is, in order to elucidate it as much as possible, we must reduce it to the particular in order to know it intuitively.”[23] I disagree here, parenthetically, with Lessing’s assumption that intuitive cognition does not require language while systematic cognition does; the function of some modes of representation is to indicate the ‘what’ that we are aware of, as well as the ‘why’ and ‘how.’ Indeed, I avoid using the word “intuition,” with its Cartesian and Kantian connotations, and prefer instead to talk about our awareness of intelligible, existing things.

Cartwright’s reading of Lessing discerns here both an epistemological claim (the general only becomes graphic or visible—I would say, thinkable—in the particular) and an ontological claim (the general exists only in the particular). Then she shows that these claims are pertinent to understanding how the theories of physics work. “First, a concept that is abstract relative to another more concrete set of descriptions never applies unless one of the more concrete descriptions also applies. These are the descriptions that can be used to ‘fit out’ the abstract description on any given occasion. Second, satisfying the associated concrete description that applies on a particular occasion is what satisfying the abstract description consists in on that occasion.”[24] In other words, abstract descriptions can only be used to say true things if they are combined with concrete descriptions that fix their reference in any given situation. We may want to ‘fit out’ a polynomial by a real-valued algebraic function, for example, or a positive whole number by a line divided into units.

This does not, however, entail that abstract concepts are no more than collections of concrete concepts. Cartwright argues, “The meaning of an abstract concept depends to a large extent on its relations to other equally abstract concepts and cannot be given exclusively in terms of the more concrete concepts that fit it out from occasion to occasion.”[25] The more abstract description of the situation adds important information that cannot be “unpacked” from any or even all of the concrete descriptions that might supplement it, or from our unreflective awareness of the thing or things successfully denoted. Likewise, the more concrete descriptions have meanings of their own that are to a large extent independent of the meaning of any given abstract term they fall under; we cannot deduce the content of the concrete descriptions by specifying a few parameters or by plugging in constants for variables in the abstract description. The relation between the abstract description and the concrete descriptions is not the same as the Aristotelian relation between genus and species, where the species is defined in terms of the genus plus some differentiating feature; nor is it captured by the relation of supervenience, nor by the relation determinable-determinate.

Here is another way of putting the insight. Abstract terms can only be used to say something true when they are combined with more concrete locutions in different situations that help us to fix their reference. And there is no "sum" of such concrete locutions. Conversely, as Leibniz argues, concrete terms can only be used to say something true when they are combined with more abstract locutions that express the conditions of intelligibility of the thing denoted, the formal causes that make the thing what it is, and so make its resemblance to other things possible. And there is no "sum" of the conditions of intelligibility of a thing.[26] We cannot totalize concrete terms to produce an abstract term and we cannot totalize abstract terms to produce a concrete term that names a thing; and furthermore we cannot write meaningfully and truthfully without distinguishing as well as combining concrete and abstract terms. So we are left with an "essential ambiguity" that results from the logical slippage that *must* obtain between the concrete and the abstract. There is an inhomogeneity that cannot be abolished, which obtains between the abstract terms that exhibit and organize the intelligibility of things, and the concrete terms that exhibit how our understanding bears on things that exist in the many ways that things exist. We need to use language that both exhibits the 'what' of the discourse, and identifies the formal causes, the 'why' of the things investigated. To do so, we need to use different modes of representation in tandem, or to use the same mode of representation in different ways, to use it ambiguously.

This is why we cannot assert the identity of something, "A = A" without using "A" in two different ways. The left-hand A, which denotes iconically something that exists, is used differently from the right-hand A, which indicates symbolically that self-sameness is a condition of the intelligibility of A, as of course it is. For our model of rationality here, we need Leibniz's notion of harmony as unity in diversity and diversity in unity, governed by the Principle of

Continuity that allows different things to stand in rational relation without denying their difference. The discussion of the transcendental curve the catenary (la chainette) in various letters exchanged between Leibniz and Huygens, as well as in two papers of Leibniz published in the *Acta Eruditorum*, will illustrate my claim. The first is “De linea in quam flexile se pondere proprio curvat, ejusque usu insigni ad inveniendas quotcunque medias proportionales et logarithmos,”[27] and the second is “De solutionibus problematis catenarii vel funicularis in actis junii an. 1691, aliisque a Dn. Jac. Bernoullio propositis.”[28] Leibniz also published an exposition of it in the *Journal des Sçavans*.[29]

4. Leibniz's Study of Transcendental Curves

Leibniz's investigations of transcendental curves do more than recast algebra, and unify geometry, number theory, and algebra in a new way; they also bring mechanics and mathematics into a new alignment. Once Newtonian mechanics is in place, differential equations become the idiom in which mechanics is translated into the Enlightenment by Leibniz, the Bernoullis, and Euler. The first transcendental curves to be studied in the mid- to late-seventeenth century were generated by tracing procedures, or defined by conditions dubbed ‘mechanical’ as a derogatory term by mathematicians who held to Euclidean or Cartesian methods. One such curve was the catenary; it is defined as the curve that a chain suspended from two fixed points assumes under the influence of gravity—and thus as the curve that allows its center of gravity to hang lowest. Galileo erroneously hypothesized that it was the parabola, but his interest in the problem brought it to the attention of others. Thus from the start there were two problems of reference associated with the catenary: it had to be distinguished from the parabola by exact criteria, and it had to be shown to exist "really" for geometry. The problems were shown by exhibiting its shape (different from that of the parabola because it is less pointed at its inflection point) in the context of symbolic explanations of the iconic shape, including the proportions of geometry, the new idiom of differential equations, and algebraic equations. The catenary, thus shown to be transcendental and not algebraic, was of special interest as a "mechanical curve" because its defining condition was apparently neither dynamical or kinematical, but rather statical and so in a less controversial sense (by seventeenth century standards) geometrical.

In the "Tentamen Anagogicum," Leibniz discusses his understanding of variational problems, fundamental to physics since all states of equilibrium and all motion occurring in nature are distinguished by certain minimal properties; his new calculus is designed to express such problems and the things they concern. The catenary is one such object; indeed, for Leibniz its most important property is the way it expresses an extremum, or, as Leibniz puts it in the "Tentamen Anagogicum," the way it exhibits a determination by final causes that exist as conditions of intelligibility for nature. And indeed the catenary, and its surface of rotation the catenoid (which is a minimal surface, along with the helicoid), are found throughout nature; their study in various contexts is pursued by physicists, chemists, and biologists.[30]

The differential equation, as Leibniz and the Bernoullis discussed it, expresses the "mechanical" conditions which give rise to the curve: in modernized terms, they are $\mathrm{d}y/\mathrm{d}x = ws/\mathrm{H}$, where ws is the weight of s feet of chain at w pounds per foot, and H is the horizontal tension pulling on the cable. Bernoulli's differential equation, in similar terms, sets $z\mathrm{d}y = a\mathrm{d}x$, where z is a curved line, a section of the catenary proportional to the weight H, and a is an appropriate constant. It can be rewritten as $\mathrm{d}y = \mathrm{a}\ \mathrm{d}x / \sqrt{(x^2 - a^2)}$. Bernoulli solves the differential equation by reducing the problem to the quadrature of a hyperbola, which at the same time explains why the catenary can be used to calculate logarithms.[31] The solution to the differential equation proves to be a curve of fundamental importance in purely mathematical terms, the hyperbolic function $y = a \cosh x/a$ or simply $y = \cosh x$ if a is chosen equal to 1.

In "De linea in quam flexile," Leibniz exhibits his solution to the differential equation in different, geometrical terms: he announces "Here is a Geometrical construction of the curve, without the aid of any thread or chain, and without presupposing any quadrature."[32] That is, he acknowledges various means for defining the catenary, including the physico-mechanical means of hanging a chain and the novel means of writing a differential equation; but in order to explain the nature of the catenary he gives a geometrical construction of it. [Diagram 1] The point-wise construction of the catenary makes use of an auxiliary curve which is labeled by points 3ξ, 2ξ, 1ξ, A (origin), 1(ξ), 2(ξ), 3(ξ).... This auxiliary curve, which associates an arithmetical progression with a geometrical progression, is constructed as a series of mean proportionals, starting from a pair of selected segments taken as standing in a given ratio D:K; it is the exponential curve. Having constructed e^x, Leibniz then constructs every point y of the catenary curve to be $1/2(e^x + e^{-x})$ or $\cosh x$. "From here, taking ON and O(N) as equal, we raise on N and (N) the segments NC and (N)(C) equal to half the sum of Nξ and (N)(ξ), then C and (C) will be points of the catenary FCA(C)L, of which we can then determine geometrically as many points as we wish."[33]

Leibniz then shows that this curve has the physical features it is supposed to have (its center of gravity hangs lower than any other like configuration) as well as the interesting properties that the straight line line OB is equal to the curved segment of the catenary AC, and the rectangle OAR to the curved area AONCA. He also shows how to find the center of gravity of any segment of the catenary and any area under the curve delimited by various straight lines, and how to compute the area and volume of solids engendered by its rotation. It also turns out to be the evolute of the tractrix, another transcendental curve of great interest to Leibniz; thus it is intimately related to the hyperbola, the logarithmic and exponential functions, the hyperbolic cosine and sine functions, and the tractrix; and, of course, to the catenoid and so also to other minimal surfaces.

An important difference between Descartes and Leibniz here is that Leibniz regards the mechanical genesis of these curves not as detracting from their intelligibility, but as constituting a further condition of intelligibility for them. As new analogies are discovered between one domain and another, new conditions of intelligibility are required to account for the intelligibility of the hybrids that arise as new correlations are forged. The analytic search for conditions of intelligibility of things that are given as unified yet problematic (like the catenary) is clearly quite different from the search for a small, fixed set of axioms in an axiomatization. The catenary is intelligible because of the way in which it exhibits logarithmic relations among numbers; and embodies the function that we call cosh, from whose shape we can "read off" its rational relation to both the exponential function and the hyperbola; and an equilibrium state in nature; and a kind of duality with the tractrix; and whatever deep and interesting aspect we discover next. Generally, we can say that the things of mathematics, especially the items that are fundamental because they are canonical, become more meaningful with time, as they find new uses and contexts. Thus the conditions of their intelligibility may expand, often in surprising ways. When the differential equation of the catenary is 'fitted out' with a geometrical curve or an equilibrium state in rational mechanics, to use Cartwright's vocabulary, the combination of mathematical representations allows us both to solve problems and to refer successfully, that is, to discover new truths.

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Notes

[1] Franklin Perkins, *Leibniz and China, A Commerce of Light*, Cambridge: Cambridge University Press, 2004, pp. 96-97.
[2] Emily Grosholz, "Was Leibniz a Mathematical Revolutionary?" in *Revolutions in Mathematics*, D. Gillies, ed., Oxford: Clarendon Press, 1992.
[3] See H. Sinaceur, A*Ars inveniendi* et théorie des modèles,@ *Dialogue* XXVII (1988), pp. 591-613); and E. Grosholz, "Theomorphic Expression in Leibniz=s *Discourse on Metaphysics*," *Studia Leibnitiana*, Thematic Issue: Towards the Discourse on Metaphysics, G. H. R. Parkinson, ed., 33, 1 (2001), pp. 4-18.
[4] See C. Imbert, *Pour une histoire de la logique*, Paris: Presses Universitaires de France, 1999.
[5] See E. Grosholz, "The Partial Unification of Domains, Hybrids, and the Growth of Mathematical Knowledge," in *The Growth of Mathematical Knowledge,* E. Grosholz and H. Breger, eds., Dordrecht: Kluwer, 2000, pp. 81-91; my notion of hybrid is discussed at length in Carlo Cellucci=s *Filosofia e matematica,* Rome: Editori Laterza, 2002, Ch. 37.
[6] *G. W. Leibniz's Mathematische Schriften, Geometrie -- Zahlentheorie -- Algebra 1672-1676* (Series VII, Vol. 1), E. Knobloch and W. Contro, eds., Berlin: Akademie Verlag, 1990.
[7] M. Parmentier, *La naissance du calcul différentiel, 26 articles des Acta eruditorum*, Paris: J. Vrin, 1995.
[8] AE Feb. 1682, GM V 118-22.
[9] MS V 392-410.
[10] MS III/1 71-3.
[11] AE June 1686; GM V 226-33.
[12] M. Parmentier, op. cit., pp. 140f.
[13] AE June 1686, MS V pp. 226-233. Parmentier, op. cit., p. 140, my translation.
[14] MS III/1, pp. 71-3.
[15] Parmentier, op. cit., p. 141, my translation.
[16] MS V, pp. 392-410. Translated in J. M. Child, *The Early Mathematical Manuscripts of Leibniz*, New York: Open Court, 1920, pp. 22-58.
[17] Ibid, pp. 49-53.
[18] R. Hendry, "Mathematics, Representation and Molecular Structure" in Klein, ed., *Tools and Modes of Representation in the Laboratory Sciences*, Dordrecht: Kluwer, 2001, p. 227.
[19] E. Grosholz and E. Yakira, Leibniz's Science of the Rational, *Studia Leibnitiana* Sonderheft 26, Stuttgart: Franz Steiner Verlag, 1998, Ch. 1.
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[22] Nancy Cartwright, *The Dappled World, A Study of the Boundaries of Science*, Cambridge: Cambridge University Press, 1999.
[23] Ibid., p. 38. Cartwright cites G. E. Lessing , *Abhandlungen über die Fabeln*, Stuttgart: Phillipp Reclam, 1759 / 1967, sec. I, p. 100.
[24] Ibid., p. 49.
[25] Ibid., p. 40.
[26] E.Grosholz and E. Yakira, op. cit., pp. 56-72.
[27] AE June 1691, MS V, pp. 243-7; translated in Parmentier, op. cit., pp. 189-199.
[28] AE Sept. 1691, GM V, 255-8; translated in Parmentier, op. cit., pp. 200-209.
[29] "De la chainette, ou solution d'un problème fameux, proposé par Galilei, pour servir d'essai d'une nouvelle analyse des infinis, avec son usage pour les logarithmes, et une application à l'avancement de la navigation. MS V pp. 258-63.
[30] "Tentamen anagogicum," PS VII, pp. 270-79.
[31] Opera Johannis Bernoulli, G. Cramer, ed., Geneva, 1742, t. III, p. 494.
[32] AE June 1691, MS V, pp. 243-7; Parmentier, op. cit., p. 193.
[33] Parmentier, op. cit., p. 194.