Chapter 7: Newton's *Principia*

Molecules cannot be perceived directly by human beings, though their appearance is inferred indirectly from traces detected by various laboratory instruments. One important feature they share with macroscopic objects is shape, and, as we saw in Chapter 3, 4, and 5, geometrical shape may thus serve an important role in linking the microscopic and macroscopic, the molecular world and the world of the laboratory, field, and classroom. Geometrical shape also serves to bring the finite realm of Euclidean geometry and the infinitary / infinitesimalistic realm of Leibniz's and Newton's new calculus (representing dynamical processes) into rational relation. When icons are used to do this kind of conceptual bridging, their significance must be carefully contextualized and explained in symbolic and natural language, not least because the import of such icons is often ambiguous. That ambiguity must be carefully constrained and exploited to be effective in the growth of knowledge. A pyramidal sketch of the molecule NH_3 refers to both the configuration of the molecule and a purified substance in a lab (which as a gas has no macroscopic shape); the roundness of a kernel of corn in McClintock's photographs refers to both the surface of a macroscopic object and the completeness of a microscopic process; the triangles SBC and SBc in the diagram to Proposition I, Book I of Newton's *Principia* [Figure 7.1] refer to both finite and infinitesimal configurations. The use of each of these images in its attendant argument depends upon and exploits its ambiguity. Ambiguity, as every philosopher knows, must be carefully managed in order to avoid confusion and contradiction; but in these arguments it is in fact successfully managed. Such successful management of ambiguity is of great philosophical interest, as I have been arguing, for a theory of knowledge in general and scientific and mathematical knowledge in particular.

Many philosophers—and mathematicians—object to the use of diagrams because they are not 'rigorous' and may be misleading when used as evidence in an argument. The objection rests, I believe, on the unfortunate Kantian assumption that images belong to intuition (construed as he construes it) in tandem with the Cartesian assumption that intuition is self-evident. Thus philosophers tend to assume that an image means only one thing and wears its meaning, as it were, on its face. When an image fails to meet this Cartesian standard (as it inevitably must) it is rejected as insufficiently rigorous. But in the case studies we have examined so far, the images are, and must be, framed by explanation in symbolic and natural language; and they are often, ineluctably, ambiguous. Moreover, some of the most important images are present in the argument because they play an indispensable role, that is, to represent shape *as shape*. Shape is irreducible. It is true that 19th century theory of functions and 20th century quantum mechanics have posited some important objects that cannot be directly pictured, though our investigation of them typically involves pictures used indirectly and by analogy. But many other objects of mathematics and physics *can* be pictured, and the use of iconic representation in their investigation is still rigorous in context. The indispensability and irreducibility of shape explains why the evolute of the ellipse (a star-shaped curve called the asteroid), the cycloid (which is its own evolute), the spiral (the involute of the circle) and the catenary (the evolute of the tractrix) are all pictured in striking diagrams at the end of Giuseppe Peano's *Formulario Mathematico*, his formalization of mathematical knowledge in the eighth and final section entitled 'Theory of Curves.' {1} Their presence is noteworthy, since Peano is often closely associated with a group of mathematicians and philosophers intent upon the arithmetical-logical formalization of mathematics. Yet Peano acknowledged that his exposition was incomplete without the presence of the diagrams, for he knew that these algebraic and transcendental curves, as shapes in space, are fundamental to mathematics.

Current philosophy of mathematics either assimilates geometry to arithmetic and then to logic, or refers it to a sense-data account of perception, a tendency that has almost completely banished the things of geometry from philosophical discourse. We have lost sight of the importance of the integrity of shape, and of geometrical form generally, an integrity that becomes clearer when we view the development of mathematical knowledge as a process of Leibnizian analysis and the things of geometry as intelligible unities in need of analysis. Before we look at Newton's great proof of the inverse square law, in which geometric shape is central and the polyvalent use of diagrams is indispensable, I will discuss the approach of an influential contemporary philosopher of mathematics, Philip Kitcher, and show why, in this case study, he cannot explain certain significant features of the proof.

1. Philip Kitcher on History

What use should the philosophy of mathematics make of history? Kitcher=s *The Nature of Mathematical Knowledge* (1983) in an admirable way intends to bring the philosophy of mathematics into relation with the history of mathematics, as a quarter of a century earlier Kuhn, Toulmin, and Lakatos aimed to do for the philosophy of science. {2} Yet Kitcher=s book seems in spirit quite ahistorical, and contains no philosophical meditation on history itself, but rather what he calls a 'defensible empiricism' opposed to Kantian apriorism and Platonism. In an earlier paper leading up to the book, he writes, 'a very limited amount of our mathematical knowledge can be obtained by observations and manipulations of ordinary things. Upon this small basis we erect the powerful general theories of modern mathematics.' {3} (This is the same general strategy of building upon a perceptual-empiricist basis employed by Penelope Maddy in her books *Realism in Mathematics* and *Naturalism in Mathematics* {4} and by Donald Gillies in his *Philosophy of Science in the Twentieth Century* and *Philosophical Theories of Probability*. {5} He continues: 'My solution to the problem of accounting for the origins of mathematical knowledge is to regard our elementary mathematical knowledge as warranted by ordinary sense perception... Yet to point to the possibility of acquiring *some* kind of knowledge on the basis of observation is not to dispose of the worry that, properly speaking, *mathematical statements* cannot be known in this way. Hence a complete resolution of the question of the origin of mathematical knowledge should provide an account of the content of mathematical statements, showing how statements with the content which mathematical statements are taken to have can be known on the basis of perception.' He does admit that 'a full account of what knowledge is and of what types of inferences should be counted as correct is not to be settled in advance...' especially since most current epistemology 'is still dominated by the case of perceptual knowledge' and restricted to 'intra-theoretic' reasoning. {6} However, his own 'epistemological preliminaries' seem nonetheless to be so dominated and restricted: 'On a simple account of perception, the process would be viewed as a sequence of events, beginning with the scattering of light from the surface of the tree, continuing with the impact of light waves on my retina, and culminating in the formation of my belief that the tree is swaying slightly; one might hypothesize that none of my prior beliefs play a causal role in this sequence of events.... A process which warrants belief counts as a basic warrant if no prior beliefs are involved in it, that is, if no prior belief is causally efficacious in producing the resultant belief. Derivative warrants are those warrants for which prior beliefs are causally efficacious in producing the resultant belief.' A warrant is taken to refer to processes that produce belief 'in the right way.' Then 'I know something iff I believe it and my belief was produced by a process which is a warrant for it.' {7} This is an account of knowledge with no historical dimension. It also represents belief as something which is caused, for a basic warrant is a causal process which produces a physical state in us as the result of perceptual experience and which can (at least in the case of beliefs with a basic warrant) be engendered by a physical process.

In Kitcher=s book *The Nature of Mathematical Knowledge*, he makes a different kind of claim about mathematical knowledge, characterizing 'rational change' in mathematics as that which maximizes the attainment of two goals: 'The first is to produce idealized stories with which scientific (and everyday) descriptions of the ordering operation that we bring to the world can be framed. The second is to achieve systematic understanding of the mathematics already introduced, by answering the questions that are generated by prior mathematics.' {8} He then goes on to propose a concept of 'strong progress,' in which optimal work in mathematics would tend towards an optimal state of mathematics: 'We assume that certain fields of mathematics ultimately become stable, even though they may be embedded in ever broader contexts. Now define the limit practice by supposing it to contain all those expressions, statements, reasonings, and methodological claims that eventually become stably included and to contain an empty set of unanswered questions.'

So there are two different kinds of assumption, in his book and the later article, that render Kitcher=s account ahistorical. One is that mathematical knowledge has its origins in physical processes that cause fundamental beliefs in us (and these processes, while temporal, are not historical). The other is that mathematics should optimally end in a unified, universal, axiomatized system where all problems are solved and have their place as theorems. This unified theory has left history behind, like Peirce=s 'end of science' or Hegel=s 'end of history,' and viewed in light of it, history no longer matters—its intervention between the ahistorical processes and objects of nature, and the ahistorical Ultimate System seems accidental. Indeed, in Kitcher=s account of rational mathematical change or 'rational interpractice transitions,' the emphasis is on generalization, rigorization, and systematization, processes that sweep mathematics towards the Ultimate System, with its empty set of unanswered questions.

2. Jean Cavaillès on History

At this point, for the sake of contrast I would like again to bring up the philosophy of mathematics of Jean Cavaillès. With Emmy Noether, he edited and translated into French the correspondence between Cantor and Dedekind; he also wrote important works on the axiomatic method, logic, and the history of set theory. The method of Cavaillès, like that of his teacher at the Ecole Normale Supérieure Leon Brunschvicg, is historical. He rejects the logicism of Russell and Couturat, but rejects as well the appeal of Brouwer and Poincaré to a specific mathematical intuition, referring the autonomy of mathematics to its internal history, 'un devenir historique original' {9} which can be reduced neither to logic or physics. In his historical researches (as for example into the genesis of set theory), Cavaillès is struck by the ability of an axiomatic system to integrate and unify, and by the enormous 'autodevelopment' of mathematics attained by the increase of abstraction. The nature of mathematics and its progress are one and the same thing for him: the movement of mathematical knowledge reveals its essence—its essence is the movement. For Cavaillès, history is a kind of discipline for the philosopher, preventing him from indulging in a discourse which is too general and finally not rigorous. It allows him to retrieve lost links, and to examine the cross-fertilization of methods and the translation of one theory into another, the transversal nature of mathematics. But Cavaillès (unlike his Hegelian teacher Brunschvicg) resists the temptation to totalize. History itself, he claims, while it shows us an almost organic unification *en acte* also saves us from the illusion that the great tree may be reduced to one of its branches. The irreducible dichotomy between geometry and arithmetic always remains, and the network of tranversal links engenders multiplicity as much as it leads towards unification. Moreover, the study of history reminds us that experience is work, activity, not the passive reception of a given. {10}

What interests me most is Cavaillès= claim that a mathematical result exists only as linked to both the context from which it issues, and that which it produces, a link which seems to be both a rupture and a continuity. {11} The 'unforeseeability' is not merely psychological, not merely subjective, not merely human, any more that the creation of the novelty is merely a human construction. The disruption, like the new creation, lies in the mathematical objects as well as in the mind of the mathematician. Thus Cavaillès writes, 'I want to say that each mathematical procedure is defined in relation to an anterior mathematical situation upon which it partially depends, with respect to which it also maintains a certain independence, such that the result of the act [*geste*] can only be assessed upon its completion.' {12} A significant mathematical act, like Descartes' solution to Pappus' problem or Newton's proof of the inverse square law discussed below, is related both to the situation from which it issues and to the situation it produces, extending and modifying the pre-existing one. To invent a new method, to establish a new correlation, even to extend old methods in novel ways, is to go beyond the boundaries of previous applications; and at the same time in a proof the sufficient conditions for the solution of the problem are revealed. What Cavaillès calls 'the fundamental dialectic of mathematics' is an alliance between the necessary and the unforeseeable: the unforeseeability of the mathematical result is not appearance or accident, but essential and originary; and the connections it uncovers are not therefore contingent, but truly necessary. Mathematical necessity is historical, but it is necessity nonetheless.

Thus, we need to learn the lessons of Hegel and Peirce without borrowing their tendencies to *totalize* the processes of history they so brilliantly addressed. As I have argued at length, the pragmatist correction of the semantic approach to philosophy of science and mathematics (itself a correction of the syntactic approach), reinstates history in philosophical reflection. Summarizing his arguments against the primary epistemic role given to the notion of 'isomorphism' in the semantical account of truth as a relation among a theory and its models, my fellow pragmatist Robin Hendry writes, 'Firstly, representation cannot be identified with isomorphism, because there are just too many relation-instances of isomorphism. Secondly, a particular relation-instance of isomorphism is a case of representation only in the context of a scheme of use that fixes what is to be related to what, and how. Thirdly, in reacting to the received [syntactic] view's linguistic orientation, the semantic view goes too far in neglecting language, because language is a crucial part of the context that makes it possible to use mathematics to represent. Natural languages afford us abilities to refer, and equations borrow these abilities. We cannot fully understand particular cases of representation in the absence of a 'natural history' of the traditions of representation of which they are a part.' {13}

Important problems often reorganize and extend mathematics via the processes by which they are solved. All of them work by indirection, and establish new correlations that transcend or traverse the boundaries of domains as they had been settled up to that point. This reorganization and extension is unprecedented, but once established the determinacy of the things correlated renders it determinate as well and far from subjective. Emerging conditions make new things and new alignments possible. And many problem-solving methods depend upon, or depart from, the canonicity of certain mathematical things vis-à-vis others. The unit in one sense and the prime numbers in another sense are canonical vis-à-vis the natural numbers; the triangle is canonical vis-à-vis a large family of surfaces; sine and cosine are canonical vis-à-vis a large family of functions; and so forth. These relations of primacy are also not 'merely subjective,' and they, like the importance of important theorems, are absent in the presentation given by formal systems invoked by Kitcher. In fact, a formal system can't explain anything; it only formalizes explanations that are brought to it. Mathematics is rational because of the way in which it furnishes explanations.

3. Book I, Propositions I and VI in Newton's *Principia*

In the *Principia* one task of the symbolic languages of the theory of proportions and (in a submerged way) algebra, and the iconic language of diagrams is to express the likeness of things that are different—geometrical things and the solar system. Moreover, the task of bringing these things into rational relation rebounds upon those languages and alters them. The theory of proportions is reconceived to include 'ultimate' ratios between elements which are infinitely or indefinitely small, 'nascent' or 'evanescent.' Algebra is pushed in the direction of differential equations, a tendency that is realized on the Continent, not in England. And the geometrical diagram is altered in its import: not only are evanescent segments and arcs introduced, but curves and surfaces are thought of as generated by various motions, each of which has its velocity or 'fluxion.' These alterations are essential to the correlation of geometrical things with time and force. As François De Gandt observes, 'The argumentation of the *Principia* remains resolutely geometric: the reasoning consists in interpreting the figure, in reading there certain relations of proportionality, which are then transformed according to the usual rules. The grand innovation, in relation to the ancients, consists in studying what these relations become when certain elements of the figure tend toward limiting positions or become infinitely small.' And a page later he adds, 'The deepest rupture is in the very definition of magnitudes or quantities: the mathematics of the ancients speaks of fixed or determinate quantities, while Newton treats of quantities that can "tend... to," "approach," up to an ultimate situation. Variation and time are here essential to the very meaning and definition of magnitude.' {14}

A related development stems from the algebraization of geometry. For geometers of the late seventeenth century like Barrow and Newton, or Schooten, Huygens and Leibniz, a curve is understood to embody relations among several variable geometrical quantities, which are defined with respect to a variable point (x, y) on the curve. These quantities include the abscissa, ordinate, arclength, tangent, subtangent, normal, radius and polar arc, as well as the area between the curve and the x-axis, and the area of the rectangle xy. {15} The relations among these quantities are represented, if possible, by equations: Leibniz's attempts to expand this kind of representation leads to the theory of differential equations and the exploration of transcendental relations. In short, the curve analyzed by algebra induces a family of related quantities constrained by the shape of the curve: it creates and expresses a system of geometrical quantities. Thus when a curve is thought of as a trajectory, it may also be thought to express a system of physical parameters which mutually constrain each other: for Newton a trajectory is the nexus of an interplay of forces. A trajectory is an odd object of thought. We are used to thinking about the shape of a trajectory, because many objects leave traces behind them as they move: a smoking torch or a jet plane leaves clouds behind it marking its path, a fish leaves a trail of bubbles in water, a boat leaves stream lines, a pen leaves a trail of ink on paper. Yet such a trace is a hypostatization, since a trajectory is the lingering and fixed record of a temporal and evanescent process; it is an intelligible object, like a melody, but a highly abstract one. Since it is so abstract and yet located precisely in time and space, a trajectory is a good locus for bringing geometry and mechanics into rational relation. For Leibniz and Newton it is not only temporal but also the record of a dynamic process that expresses the interplay of forces.

By examining how natural and formal languages link geometry and the solar system, I hope to shed light more generally on how mechanics brings mathematics into rational relation with physical reality (material or perceptible or phenomenal). This is not the same as bringing it into relation with the 'given,' *pace* Descartes, Locke, Kant, and A. J. Ayers. Recall that Descartes treats 'res extensa' as the datum in his physics; Locke treats 'ideas of sense' as data that ground his philosophical defense of science, A. J. Ayer revives that empiricist notion in the 'sense data' that ground his phenomenalism; and Kant treats the manifold of sensible intuition as the given in his account of science in the first *Critique*. All of them treat the given as a surd that must be worked up by the mind, organized and unified, before it can be thought. By contrast, I side with Leibniz and the principle of reason in holding that anything that might be merely 'given' would be by its very definition unintelligible and so could not be encountered as an existing whole. Otherwise put, there is no merely material or perceptible stuff or a 'manifold of sense' that thought must then organize. Moreover, structure itself—the structure expressed by language—is intelligible only in relation to existing things, things that express the structure. (To express structure is not the same as to instantiate it; instantiation is a notion special to logic that deserves closer scrutiny within current philosophy of logic and should only very cautiously be exported.) Structure cannot be thought by itself. Briefly stated, I am trying to avoid the myth of the given as well as the myth of pure syntax. {16}

Proposition XI in Book I of the *Principia* forges a novel relation between the solar system and geometry by explaining why the ellipse *qua* trajectory is a condition of intelligibility for the stably persistent solar system, and the process of explanation registers important changes in mechanics as well as mathematics. {17} The proposition builds on a process that is already more than a century old and involves the reconstruction of understanding the earth and the sun, and of geometry, in the work of Copernicus, Tycho Brahe, Kepler, Descartes, Galileo, Huygens, and others. To understand the genealogy of Proposition XI, we must go back a bit in Book I, to Proposition I (which shows how to represent time by means of geometry) and Proposition VI (which shows how to represent force). Proposition I is Newton's generalization of Kepler's law of areas: 'The areas which revolving bodies describe by radii drawn to an immoveable center of force do lie in the same immoveable planes, and are proportional to the times in which they are described.' {18} (Recall that Kepler's formulation of this law, like his claim that planetary orbits are elliptical, rests both on his sense of geometrical propriety and on the vastly improved astronomical data that he inherited from Tyco Brahe.) Newton's proof of Proposition I is accompanied by a diagram [Figure 7.1] where S is the center of force. A body proceeds on an inertial path from A to B in an interval of time; if not deflected, it would continue on in a second, equal interval of time along the virtual path B*c*. However, Newton continues, 'when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse' so that the body arrives not at *c*, but at C. Then *c*C (= BV) represents the deflection of the body due to the force; indeed, as will become apparent, cC = BV becomes the geometrical representative of the force. The perimeter ABCDEF... is the trajectory of the body as it is deflected at the beginning of each equal interval of time by discrete and instantaneous impulsions from S. Newton then uses the Euclidean theorem that triangles with equal bases and equal elevations have equal areas, to show that the area ΔSAB = area ΔSB*c* = area ΔSBC; this equality extends to triangles SDC, SED, SFE... by the same reasoning, so that equal areas are described in equal times. We have only, as Newton says, 'to let the number of those triangles be augmented, and their breadth diminished in infinitum' for this result to apply to a continuously acting force and a curved trajectory.

Notice that this result holds for any kind of central force. The regularity of the sweeping out of areas is linked only to the directionality of deviation; if the deviation Cc is always directed towards a fixed point, then the areas swept out are proportional to the time. Force is defined only as the ability of a 'center of force' (whatever that is) to cause a body to deviate from inertial motion. Thus it appears that the trajectory of a planet, whatever its mathematical form, in turn has a condition of intelligibility that is defined formally and causally: a center of force that leads to its deflection in a regular way from uniform motion in a straight line. The shape of the trajectory testifies to the center of force and the law it obeys; the lawful center of force explains the shape of the trajectory.

The diagram and its accompanying proportions are thoroughly ambiguous, and must be read in two incompatible ways. Read as a collection of finite line segments and areas, where the perimeter is a polygon, they allow for the application of Euclidean theorems to the problem; read as a collection of infinitesimal as well as finite lines and areas, where the perimeter is a curve, they become pertinent to accelerated motion, time, and force. Thus the diagram, whose meaning and intent cannot be understood unless it is read in both ways, could not have arisen in Euclidean geometry; indeed, even if construed as a finite configuration, it could not have arisen there or in the geometry of Archimedes. It does not even resemble a classical problem of quadrature, for if it were the perimeter would be a known curve, there would be no reason to single out the point S, or to study the line segments B*c* and *c*C = BV. The diagram makes sense only with reference to a problem about force, time, and motion, in which geometry enters as an auxiliary means to help solve it. What Newton finds in Kepler's law of areas is a way to express time in terms of non-uniform rather than uniform motion, as the sweeping out of equal areas in equal times, and a way to identify in formal terms a center of force.

Proposition XI is a special case of the general result that Newton works out in Proposition VI, where he shows that for any kind of revolution APQ of a body P around a center of force S, the centripetal force will be inversely proportional to the quantity SP^2 x QT^2 / QR. [Figure 7.2] Newton writes, 'In a space void of resistance, if a body revolves in any orbit about an immoveable center, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn bisecting the chord, and produced passing through the center of force: the centripetal force in the middle of the arc will be directly as the versed sine and inversely as the square of the time.' {19} The deviation QR is proportional to the intensity of the force tending to S, and also proportional to the square of the time.

The proof runs as follows. PR is the virtual trajectory the body would have followed if it had not been deflected by S, and by the First Law that governs inertial motion it is directly proportional to time *t*. By Proposition I, the curvilinear area SPQ is also proportional to *t*; and since in 'the least time' it may be considered a triangle, 2 SPQ = SP x QT. Then PR, proportional to *t*, is also proportional to SP x QT, an evanescent area. The segment QR represents the virtual deviation of the body as a result of an 'impulse' of force in 'the least time,' and is thus directly proportional to the force. It is also directly proportional to t^2, by Lemma X: 'The spaces which a body describes by any finite force urging it... are in the very beginning of the motion to each other as the squares of the times.' Here Newton asserts an analogy between any such force, and gravity, generalizing Galileo's result that in free fall the space traversed is proportional to the square of the time—in the first instant of motion, 'the very beginning of the motion.' In sum, F is proportional to QR / SP^2 x QT^2. In terms of the diagram and Newton's preference for writing his result so that the ratio has three dimensions, the force is inversely as SP^2 x QT^2 / QR: 'the centripetal force will be inversely as the solid SP^2 x QT^2 / QR, if the solid be taken of that magnitude which it ultimately acquires when the points P and Q coincide.' In the sequence of propositions that follow, Newton exploits the peculiar geometrical properties of various possible trajectories APQ to transform this latter expression into another expression containing only constants multiplied by the distance SP raised to a certain power. That is, Newton explores how the geometry of the curve may characterize the force; and this is what leads to Proposition XI.

Note that very little in either the diagram of Proposition VI or the reasoning about it is drawn from Euclidean geometry: only that SP is a line segment, PY is the tangent to the curve at P, and the area of a triangle is half the product of its base and altitude. (And indeed the application of the latter theorem is quite un-Euclidean.) The exploitation of the geometry of the trajectory comes later, as Proposition VI is applied to various cases, only one of which is 'real.' The meaning of the diagram is determined for the most part by the way it represents a physical situation, since why PR, QR, and SP x QT are chosen and how they are related can only be explained by theorems of mechanics developed by Kepler, Galileo, Descartes, and Newton. Of course, these theorems also geometrize mechanics in the sense of discovering geometrical forms as conditions of intelligibility for physical things in novel ways. And Newton's way of expressing his result, 'the centripetal force will be inversely as the solid SP^2 x QT^2 / QR' allows him to exploit the proportion idiom of the Eudoxian tradition to relate and yet discriminate heterogeneous physical magnitudes, lines, and areas, and finite and infinitesimal magnitudes.

4. Book I, Proposition XI in Newton's *Principia*

In Proposition XI, Newton applies Proposition VI to the case where the trajectory is an ellipse, the form that Kepler brought to light but could not explain or put together with his law of areas. Clearly, this is a crucial step in the application of Newton's results to the System of the World in Book III. The diagram of Proposition XI [Figure 7.3] combines the physical-geometrical schema of the diagram of Proposition VI with the 'pure geometry' of the ellipse, but it is instructive to examine the combination in detail. The latus rectum L = 2 BC^2 / AC and the diameters of the ellipse, BC, SA, DK, and PG have no physical import; but the perimeter APBDGK is also the orbit of a revolving body, S also the center of force, SP also the distance of the revolving body to the center of force, and PR, QR, and SPQ = 1/2 (SP x QT) retain their physical significance. By contrast, the auxiliary lines PH, IH, and PF, like the ellipse's diameters, enter the reasoning only insofar as they are geometrical. What the figure depicts is thus a thorough hybrid, a creature of both geometry and mechanics.

The proof proceeds by establishing proportions between the segments and products of line segments by means of theorems about similar triangles, isosceles triangles, and ellipses, and culminates in an elaborate, two-sided compounding of these ratios. Newton directs, 'Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Q*v* in *x*; and complete the parallelogram Q*x*PR,' and then begins by proving that EP = AC, using auxiliary lines HI and HP (H is the other focus, besides S). Since ΔIHS is similar to ΔECS, SE = EI and EP = 1/2 (PS + PI), or 1/2 (PS + PH), since ΔPIH is isosceles. PS + PH = 2AC by the nature of ellipse construction, so EP = AC. {20}

Then Newton sets out to establish certain extended proportions—L x QR:L x P*v* :: QR:P*v* (where the constant latus rectum L = 2 BC^2 / AC). Since this refers to a physical situation that is 'nascent,' Newton is profiting from the potential openness of ':' and '::' to relate non-Archimedean magnitudes. QR (= Px): P*v*:: PE : PC because ΔP*xv* is similar to ΔPEC, another highly non-Euclidean application of a Euclidean theorem. Finally, PE : PC :: AC : PC because PE = EP = AC, so that L x QR : L x P*v* :: AC : PC.

Next, Newton asserts that L x P*v* : G*v* x P*v* :: L : G*v* and that G*v* x P*v* : Qv^2 :: PC^2 : CD^2, a fact about ellipses, except that P*v* and Q*v* are infinitesimal magnitudes. It is only at this point in the proof that Newton says explicitly, let q → p: 'when the points P and Q coincide, Qv^2 = Qx^2.' One might then take the foregoing reasoning to be about very small finite magnitudes, so that the application of Euclidean magnitudes is straightforward. However, as we shall shortly see, the final compounding of ratios even-handedly combines ratios established before and after this step in the proof. Newton uses the ambiguity of the diagram with its accompanying proportions: read as finite, it allows the application of Euclidean results; read as 'nascent,' it provides a mathematical schema for force, time, and accelerated motion.

When points P and Q coincide, Newton claims that $Qv^2 = Ax^2$ and so Qx^2 (= Qv^2) : QT^2 :: EP^2 : PF^2, since the infinitesimal triangle Q*x*T is similar to Δ PEF. Then EP^2 : PF^2 :: CA^2 : PF^2 by the first result, and CA^2 : PF^2 :: CD^2 : CB^2 by a previously established result about ellipses. Thus Qx^2 (= Qv^2) : QT^2 :: CD^2 : CB^2. Newton is now ready to carry out the final compounding, which may be summarized in the following perspicuous array. {21}

L x QR	: L x P*v*	:: AC	: PC
L x P*v*	: G*v* x P*v*	:: L	: G*v*
G*v* x P*v*	: Qv^2	:: PC^2	: CD^2
Qv^2	: Qx^2	:: 1	: 1
Qx2	: QT^2	:: CD^2	: CB^2

Edith Sylla, in her article just cited, notes that Newton has set up the left-hand ratios as a continuous series, and compounds them according to the Eudoxian tradition, taking the extreme terms and forming the new ratio L x QR : QT^2. The right-hand ratios he compounds according to the medieval tradition, by multiplication: AC x L x PC^2 x 1 x CD^2 : PC x G*v* X CD^2 x 1 x CB^2 or (substituting 2 BC^2 / AC for L and cancelling) 2PC : G*v*.

The foregoing discussion of Newton's analysis explains why Newton makes this distinction in his compounding. The magnitudes on the left-hand side have physical import and are evanescent, that is, they are just the kind of magnitude which should be treated in a manner that respects the heterogeneity of terms and—given the indeterminacy of the sign : —allows for the manipulation of infinitesimal magnitudes. Those on the right-hand side, if we recall that Gv = GP as Q → P, are all constant finite geometrical line lengths with no physical import. Having no reason to think of them other than as numbers, even when they are squared, Newton handles them according to the second tradition, multiplying them as rational numbers. The compounding thus yields the proportion, L x QR : QT^2 :: 2PC : Gv. As Q → P, 2 PC = Gv, so L x QR : QT^2 :: 1 : 1, so in turn L x QR is proportional to QT^2. Multiplying both these terms by SP^2 / QR, we find that SP^2 x QT^2 / QR is proportional to L x SP^2, or, since L is a constant, to SP^2. Thus, the central force in this problem is inversely proportional to the square of the distance SP.

Nature presents us with shapes that command our attention and invite explanation. The shape of a hanging chain is immediately visible; the path of the sun through the daytime sky is—rather slowly—visible as the ecliptic and the path of the planets through the night sky is visible too, in a more constructed and indirect sense, given the way they retrogress. Leibniz's analysis discovers the catenary (the function cosh x) as a condition of intelligibility of the hanging chain, when it is understood as an equilibrium of forces and expressed in terms of his novel, revised algebra, both as a differential equation and as the family of solutions to that equation. Newton's analysis discovers the ellipse as a condition of intelligibility for the solar system; but the ellipse serves as such a condition only in tandem with another condition of intelligibility, the formal cause of a center of force. This yoking means that the ellipse must be thought not only as a geometrical unity (which it is, and does not cease to be) but also as the trajectory of a moving body constrained by a central force. Proposition XI on the one hand, and Propositions XXXIX - XLI on the other exhibit how these two conditions, one stemming from classical geometry and the other from a radically novel conception of mechanics, determine each other. The diagrams in the latter set of propositions, by the way, show how geometrical diagrams may be used in a way that is mostly symbolic and only barely iconic. Icons, especially icons that must be read in two or three different ways, do not wear their meaning on their faces; the interpretation of icons is not direct or ‘intuitive.’ And a geometrical figure can be used not as an icon but as a symbol.

In one sense analysis tries to find simples vis-à-vis a complex thing; sometimes this means finding the parts of a whole, but of course meriology takes many forms and not all analysis is meriology. Newton doesn't identify individual heavenly bodies as the simples of the solar system, but rather the subsystem of a planet and the sun and a body in inertial motion deflected in an orderly way by a center of force. The relation of this subsystem-nomological- machine, and model, to the System of the World is far from additive, unlike the relation of bricks to a wall, and the reconstitution of the whole from the simples is both difficult and in principle incomplete. The simples of the model, exhibited by the diagram-proof and the 'perspicuous array' given above, are the ratios and proportions associated with the elements of the ellipse. Here the simples are not parts in any sense. In particular, Newton explores what becomes of the important ratio $SP^2 \times QT^2 / QR$ in the case where the trajectory is an ellipse, and discovers that it is the constant $L \times SP^2$, which is the key to the whole problem. In brief, Newton reduces the complexity of the situation by treating time, force, accelerated motion, uniform motion, physical distance and merely geometrical distance evenhandedly as magnitudes and then considering the proportions in which they must stand to each other. Some of the heterogeneity amongst these terms is reinstated, as we have seen, when the ratios are compounded. Another nice example of Newtonian analysis is the result from Proposition I used in Proposition XI where two virtual (and indeed evanescent) trajectories, compounded in accordance with Corollary I of the Axioms of Motion and understood as inertial motion and as the first moment of free fall, yield the actual motion of the revolving body. So too is the way the mathematical form of the ellipse and the mechanical form of a center of force become, as the means by which geometry is brought into alignment with the solar system, conditions for the intelligibility of the world. Analysis searches for requisites, that without which the thing could not be what it is, or in mathematics, that without which a problem could not be solved. The wholes that present themselves as intelligible existents always require analysis, though they can never be completely reconstituted or retrieved from the elements that analysis discovers. This is true of the Euclidean triangle, and Newton's System of the World, and the solar system in which we find ourselves.

Chapter 7 Notes

1 G. Peano, *Formulario mathematico* (Turin: C. Guadagnini, 1894; repr. Roma: Edizioni cremonese, 1960).

2 (New York: Oxford University Press, 1983). I wrote a review essay of this book in *The British Journal for the Philosophy of Science*, 36 (1985), pp. 71-8.

3 Ibid, p. 92.

4 (Oxford: Oxford University Press, 1900); (Oxford: Oxford University Press:1997).

5 Gillies, *Philosophy of Science in the Twentieth Century*; (London: Routledge, 2000).

6 Kitcher, *The Nature of Mathematical Knowledge*, 96-97.

7 Ibid., 18.

8 Ibid., 530-1.

9 'Réflections sur le fondement des mathématiques,' *Travaux du IXe Congrès international de philosophie*, t. VI / no. 535 (Paris: Hermann, Cavailles 1937) 136-139.

10 Sinaceur, *Jean Cavaillès*, Ch.1.

11 'La pensée mathèmatique,' Bulletin de la Société française de philosophie 40 / 1, 1-39.

12 Ibid., 9.

13 Hendry, 'Mathematics, Representation and Molecular Structure,' 227.

14 De Gandt, *Force and Geometry in Newton's* Principia, 225-226. The influence of this book is apparent throughout my exposition here. I would also like to recommend the philosophical treatment of Newton in Michel Blay's *Les raisons de l'infini: Du monde clos à l'univers mathématique* (Paris: Gallimard, 1993), translated by M. B. DeBevoise into English as *Reasoning with the Infinite: From the Closed World to the Mathematical Universe* (Chicago: University of Chicago Press, 1998); and in Marco Panza's *Isaac Newton* (Paris: Les Belles Lettres, 2003).

15 See the opening pages of H. J. M. Bos, "Differentials, higher-order differentials, and the derivative in the Leibnizian calculus," *Archive for History of Exact Sciences*, 14, pp. 1-90.

16 For a deep and suggestive account of this issue, see Dale Jacquette, 'Intentionality and the

Myth of Pure Syntax**'**, *Protosoziologie*, 6, 1994, 76-89; 331-333.

17 Sir Isaac Newton's *Mathematical Principles of Natural Philosophy and his System of the World*, tr. A. Matte, ed. F. Cajori (Berkeley: University of California Press, 1934) Vol. 1, 56-57. Hereafter referred to as *Principia*.

18 Ibid., 40-42.

19 Ibid., 48-49.

20 For a more detailed account of the proof, see my 'Some Uses of Proportion in Newton's *Principia*, Book I,' *Studies in History and Philosophy of Science* 18 / 2 (1987), 209-20.

21 See E. Sylla, 'Compounding Ratios,' 16.