These comments stem from a discussion (07 and 08 / 08) with Emiliano Ippoliti, University of Rome, concerning a useful remark by Dirk Schlimm that we need an better taxonomy of axiomatic systems.

One thing that needs clarification is what “axiomatics” means.

Euclid axiomatizes geometry, putting an array of previously scattered results into a more perspicuous organization, ordered in terms of axioms, definitions and “common notions.” But the work is written in Greek, with special notation (and, perhaps, diagrams) interspersed. Euclid’s *Elements* is an organized subject matter; there is no distinction between theory and model.

Axiomatization in the modern sense means not only re-organizing a body of results under a set of axioms (some of which are logical, some of which are special to the domain), but of re-writing all the results in a formal language, for example, a first order or second order theory in predicate logic, with special vocabulary. Thus axiomatization in the modern sense requires translation, from the original mathematical idiom (usually a combination of notations, with natural language) to a formal language. Then we have a formal theory, and the model it is designed to characterize. If the formal language is well enough behaved, it will provide a mechanical procedure that can decide whether a string of wffs ending in a distinguished wff (translating a mathematical result in the model) is a proof of the translation of that result; but it will not “capture” the model, because there will be other models that also satisfy the theory.

Boole cannot be carrying out an axiomatization in the modern sense, because predicate logic was not available to him. He is also not re-organizing a body of results like Euclid either. Rather, he has an algebraic structure in hand, which works for the natural numbers / integers / algebraic numbers, and he is trying to use it to characterize the subject matter of logic—sets in one sense or propositions in another. His hypothesis (that the algebra of arithmetic will work for logic) is disappointed; he puts artificial constraints on the operations, which limit their range. The happy outcome is the insight (which Boole himself never really sees) that there can be more than one algebra, and that the algebra of sets (or of propositions) is different from the algebra of arithmetic. Boole is trying to bring a subject matter (logic) into alignment with an algebraic structure; and this task is harder than he thought it was. It requires an adjustment of the algebra, and an adjustment of the subject matter.

The task of applying an abstract algebraic structure to a problem in a mathematical domain (which means identifying which structure to choose, and how to interpret it so that it does useful work) is different from the task of axiomatization in the modern sense. To identify pertinent algebraic structures would be to engage in “axiomatics” only if one means by the term something different from Euclid or Russell.

A system that is strictly axiomatized will not allow ampliative reasoning, by definition. Mathematicians, however, are usually not interested in axiomatized formal languages, except for two reasons.

1) Mathematical logicians study the properties of formalized systems as objects of mathematical interest in their own right. The reasoning *about* such objects may well be ampliative; Goedel's theorems, for example, bring logic into novel relation with number theory in a way that is certainly ampliative.
2) Other mathematicians may be interested in formalized systems that represent (for example) number theory or differential geometry because those representations shed some light on problems that interest them. However, the reasoning is ampliative because, once again, the formalized system is brought into novel relationship with another subject matter.

As Emiliano noted, real discovery and ampliativity take place when you go beyond the domain you are working in; by definition, a formal, axiomatized system will not allow you to do this. Such a system only makes explicit information that is only implicit in the acioms, but it cannot introduce novelty.

Apropos the taxonomy of different kinds of axiomatization:

I think we need at least three sets of binary distinctions.

Concrete vs. abstract axiomatization. Euclid viewed what he was doing as bringing a set of formerly scattered results into systematic relationship, so that mathematical research would advance, and so that mathematical results could be transmitted better by teachers. This is a good idea of concrete axiomatization. By contrast, abstract axiomatization approaches a domain with intentions (usually of a philosophical or logical nature) that lie outside that domain. For example, Hilbert was obviously not a practicing Greek geometer; he took up Euclid's geometry with the intention of clarifying its inferential structure as part of his formalist program; that is abstract axiomatization.

Categorical vs. non-categorical axiomatization. Some axiomatizations, like Peano's axiomatization of arithmetic, are intended to capture all (and only) the truths of a certain domain and thus to characterize the items of that domain. Others, like the axioms that define general topology or group theory, are intended to be non-categorical. In problem-solving situations, they must be supplemented by more specific information about the items that are correctly characterized as e.g. topological spaces or groups.

Informal vs. formal axiomatization. Peano's axiomatization of number theory is informal. Russell and Whitehead take that axiomatization and translate it into the language of logic; they translate it into a formal language, which makes it a formal axiomatization. Once this occurs, the formal axiomatization can be studied on its own as a mathematical item, which is of course not true of the informal axiomatization.

One might argue that we should call some of these distinctions different *aspects* of axiomatization rather than different *kinds* of axiomatization. But that argument depends on the technical notions of a formal theory and its models, which are only pertinent to formal axiomatization. (Concrete, abstract, categorical, non-categorical, and informal axiomatizations may all be—and in mathematical practice usually are—not formal, that is, they are not given in the language of some n-order predicate logic.) Moreover, by giving the examples from the history of mathematics that I indicate above, I want to show that these distinctions are independent of the subjective intentions of the people using the axiomatization. To say that all mathematicians use group theory as a non-categorical axiomatization is an objective feature of what it is and how it may be used in mathematics. To say that Peano's and Hilbert's axiomatizations are informal is an historical fact, and so is the claim that Euclid organizes geometry from within and Hilbert organizes it from without.

One might say, Euclid doesn't seem to distinguish between theory and model. But when we read his work we can distinguish between them. I would say rather that we can now talk about non-standard models of Euclidean geometry as well as non-Euclidean geometries because the conception of geometry in relation to axiomatization has been transformed. Historically, it was transformed by the Cartesian-Leibnizian discovery of the usefulness of regarding mathematical notation as partially detachable from the items it represents, by the work of the geometers who (aided by advances in analysis) articulated the possibilities of non-Euclidean geometries, and by formalism and logicism, philosophical-mathematical approaches that employ very different conceptions of axiomatization.

The issue of whether a problem can be solved within or only outside a given axiomatization may be an objective matter. If you take Euclidean geometry to be restricted to constructions by ruler and compass, then clearly the squaring of the circle lies outside its bounds.

Many problems are solved in an area that may be axiomatized in some sense, by annexing another area of research in some of the many ways that such annexation takes place. Then the result arrived at does not lie within the original axiomatization. The proofs in modern number theory that I am familiar with are all like this; a thought experiment about a non-trivial proof in number theory that stays "within" the axiomatization of the Peano postulates seems so unlikely to me that I can't really entertain it. I don't think a practicing number theorist would know what you mean by "purely number theoretic means," but then it depends on what you mean by that phrase.

By contrast, the fact that it is difficult for someone to find a proof in predicate logic, or to make explicit what is implicit in a set of axioms seems to me truly subjective: some people are cleverer than others at non-ampliative explication, independent of what lies inside or outside of that axiomatization.

I'm interested in claims of ampliative reasoning that are objective (and historical) as opposed to subjective.