Chapter 10: Logic and Topology

Logic is usually presented as a canon of argument, a codification of the rules of thought. It formalizes the ways in which we move from premises that present evidence to the conclusions they support, and exhibits the ways that are most reliable because they transmit truth, or perhaps high probability. Understood in this way, logic has no special subject matter; thinking can be thinking about anything. However, from the very beginning of the Western tradition of logic, it has been clear that some patterns of reasoning resist formulation in, for example, Aristotle's syllogistic, which is a logic of terms: reasoning about propositions, relations, temporal succession, modalities, counterfactuals, and individuals, to name a few.

Modern predicate logic, which begins with the work of Gottlob Frege taken over by Bertrand Russell and Alfred North Whitehead, is not one system of higher order logic but rather an open-ended family of theories. Each theory consists of axioms couched in logical and extra-logical vocabulary, and 'purely logical' rules of inference to specify the licit pathways from axioms to theorems. Thus, reconstructions of arithmetic in terms of predicate logic have special terms like '0' and '1' and the successor operator S, as well as operations like '+' and 'x'. These special terms have an ambiguous status when considered as part of logic, since logic is not supposed to be tied to any special subject matter. Bertrand Russell claimed to show that the special vocabulary of arithmetic could in fact be rewritten in 'purely logical' terms. {1} Then, the reductionist might argue with Charles Chihara, arithmetic is also shown to have no subject matter. {2}

By contrast, I would argue that what emerges here is a second role for logic. It becomes a collection of not only rules of thought, but also representations for other mathematical objects and procedures. In the latter role it exhibits the deductive structure of special theories (in model theory) as well as the fine detail of definitions (in definability theory, examining the logical complexity of different mathematical items.) This added role, however, can lead to a philosophical misunderstanding, the logicist misreading of Leibniz. Russell and Couturat read Leibniz as if he claimed that a single 'universal characteristic' would prove to be the optimal representation of mathematical things. But Russell's elaborate formulae for the natural numbers in I.II.A of *Principia Mathematica* are mathematically inert, 'lifeless,' to use Angus Macintyre's adjective. *Inter alia*, they do not exhibit their most important feature (most important to number theory at any rate), their unique decomposition into prime factors, any more than the stroke notation does, and for the same reason: it lacks the economical compound periodicity of Arabic numerals. Different modes of representation in mathematics bring out different aspects of the items they aim to explain and precipitate with differing degrees of success and accuracy. Moreover, Russell's definitions of 0, 1, and 2 and so forth as logical formulae also define each number as a set of sets (so the unit 1 is, for example, the set of all one-membered sets): there are two problems with this. The first is that the definition presupposes the availability of what it defines, that is, the unit; and the second is that the set of all one-membered sets is not a set. The mathematical success of Russell's definitions, it seems to me, was negative; its incoherence led to deeper mathematical and philosophical reflection on the nature of sets.

Viewed in this representative role, first order predicate logic does in fact have an affinity with a special subject matter, depending on how we understand 'aboutness': that subject matter is sets of integers on the one hand, or recursively defined well-formed formulae on the other hand. First order predicate logic represents the universe of discourse of any of its special theories as if it were a set of discrete things with no pertinent internal structure; these things may then be collected into sets (represented by predicates) and then into increasingly 'logically complex' sets by projection and complementation using the quantifiers, recorded by the increasing complexity of the well formed formulae. In sum, first order predicate logic is not very good at representing the natural numbers themselves; it is better at representing sets of integers in one sense or well formed formulae in another, and exhibiting something useful about those sets or formulae. Its representation of the natural numbers, we might say, is highly symbolic and not allied in any useful way with more iconic modes of representation of them; while its representation of sets of integers and well formed formulae is relatively iconic. This reflection makes it seem like an irony that the key to the greatest meta-mathematical result of the last century was the representation of the well formed formulae of first order predicate logic by the natural numbers, a representation whose success depends on the unique prime decomposition of the latter.

1. Penelope Maddy on Set Theory

There are at least two reasons why there cannot be a complete speech about mathematical things, any more than there can be a complete speech about human actions. (And this is no more a reproach to the reality of mathematics things than it is to the reality of who we human beings are and what we do.) One is that there are many different kinds of mathematical things, which give rise to different kinds of problems, methods, and systems. The other is that mathematical things are investigated by Leibnizian analysis, a historical process in which some things (or features of things) that were not yet foreseen are discovered, and others forgotten. Indeed, these two aspects of mathematics are closely related. For when we solve problems, we often do so by relating mathematical things to other things that are different from them, and yet structurally related in certain ways, as when we generalize to arrive at a method, or exploit new correlations. We make use of the internal articulation or differentiation of mathematical domains in order to investigate the intelligible unities of mathematics. To put it another way: just as there is a certain discontinuity between the conditions of solvability of a problem in mathematics and its solution (as Cavaillès noted), so there is a discontinuity between a thing and its conditions of intelligibility (as Plato noted). An analysis results in a speech, which both expresses, and fails to be the final word about, the thing it considers.

But perhaps I have been too restrictive by discussing number theory, geometry and topology, and should rather consider the claims of a more encompassing theory, like set theory.

Penelope Maddy=s defense of set theory as a foundation for mathematics in the first section of Part I.2 of her *Naturalism in Mathematics* is especially instructive, for she renounces the traditional reductive claims of set theory, in favor of a defense that cites its usefulness as a resource in problem-solving. 'The identification of the natural numbers with, say, the finite von Neumann ordinals is not claimed to reveal their true nature, but simply to provide a satisfactory set theoretic surrogate; thus, no problem arises from the observation that more than one satisfactory set theoretic surrogate can be found. Likewise, the identification is not understood as a prelude to the repudiation or elimination of the original mathematical objects; though they may well drift off into irrelevancy for the most part, no such strong ontological conclusion is drawn.' {3} So the benefits of set theory are not ontological; Maddy also concedes that they are not epistemological—set theory is not provably consistent, and is not 'more certain' than the other branches of mathematics for which it provides foundations.

By contrast, Maddy goes on to argue that the benefits of set theory are 'mathematical rather than philosophical.' Granting that the heterogeneity of mathematical domains is essential to research, she writes, 'I think it cannot be denied that mathematicians from various branches of the subject—algebraists, analysts, number theorists, geometers—have different characteristic modes of thought, and that the subject would be crippled if this variety were somehow curtailed.' {4} She also invokes the analogy with the sciences, as I have done: 'Consider the claim that everything studied in natural science is physical; it doesn=t follow from this that botanists, geologists, and astronomers should all become physicists, should all restrict their methods to those characteristic of physics. Again, to say that all objects of mathematical study have set theoretic surrogates is not to say that they should all be studied using set theoretic methods.' {5}

Thus, Maddy defends set theory by pointing out that it serves two functions. First, it helps us to generalize and systematize, like abstract algebras, logic, or category theory. Second, when objects from other domains are correlated with sets, we can solve formerly unsolved problems about those objects. And these claims are true. But I think that Cavaillès would dispute—as would I—her inference from them: 'The force of set theoretic foundations is to bring (surrogates for) all mathematical objects and (instantiations of) all mathematical structures into one arena—the universe of sets—which allows the relations and interactions between them to be clearly displayed and investigated. Furthermore, the set theoretic axioms developed in this process are so broad and fundamental that they do more than reproduce the existing mathematics; they have strong consequences for existing fields and produce a mathematical theory that is immensely fruitful in its own right. Finally, perhaps most fundamentally, this single, unified arena for mathematics provides a court of final appeal for questions of mathematical existence and proof: if you want to know if there is a mathematical object of a certain sort, you ask (ultimately) if there is a set theoretic surrogate of that sort; if you want to know if a given statement is provable or disprovable, you mean (ultimately), from the axioms of the theory of sets.' {6}

Maddy's admission that mathematics 'would be crippled if this variety [the variety of its domains] were somehow curtailed' does not sit well with her claim here that the translation of their objects and methods into a set theoretic idiom 'allows the relations and interactions between them to be clearly displayed and investigated.' In most cases, such a translation would paralyze research. In a few cases, correlating numbers with set theoretic entities is helpful for solving problems, but only when both the number and its set theoretic analogue are both present in the problem-solving situation. In many cases, using set theory as a generalizing structure is very helpful, as when a number theorist considers the set of all the natural numbers; but this is not a translation of the situation into set theoretical terms—rather it is the integration of set theory into a number theoretical situation. Number theory can be generalized and reorganized by the superposition of the algebra of arithmetic, various abstract algebras, topology, set theory, and category theory, to name a few; but in this line-up set theory has no special pride of place. The unifying and clarifying virtues of various generalizing structures have all had their champions: abstract algebras (Boubaki), predicate logic in the context of model theory (Robinson), category theory (Lawvere). In this company, Maddy=s claim that set theory is 'the ultimate court of appeal' seems unsubstantiated.

2. A Brief Reconsideration of Arithmetic

Iconic representations are supposed to resemble what they represent; this is usually taken to mean that they resemble things in a visual and spatial manner. I have argued that iconic representations in geometry, as in chemistry, are typically combined with symbolic representations and natural language instructions that explain the meaning of the icons, and to which the icon lends meaning, at the level of syntax, semantics, and pragmatics. In many cases, the iconic representation is indispensable. This is often, though not always, because shape is irreducible; in many important cases, the canonical representation of a mathematical entity is or involves a basic geometrical figure. At the same time, representations that are 'faithful to' the things they represent may often be quite symbolic, and the likenesses they manifest may not be inherently visual or spatial, though the representations are, and articulate likeness by visual or spatial means. Now I would like to raise the question, what might iconic representation mean in arithmetic? What does it mean to 'picture' a number? What does it mean to 'picture' a number system? At first it might seem as if the representation of numbers is never iconic, but this rather Kantian assumption is not justified, and I counter it by showing that there are degrees of iconicity in the representation of numbers.

A natural number is either the unit or a multiplicity of units in one number. The representation of such a unified multiplicity is more iconic when the representation itself involves multiplicity: ///// is more iconic than '6' or 'six', because ////// exhibits the multiplicity of the number 6—its multiplicity can be 'read off' the representation. In the case of six strokes, the unity of the number six is indicated (iconically) by the way the six strokes are grouped together and isolated from other things on the page. Six strokes scattered around the page or the chapter would not be any kind of representation of the number 6, unless accompanied by special instructions saying how to find them and *then* to put them together (in thought or concretely). The putting together and isolating is an aspect of graphic representation that stands for the intelligible unity of the number: natural numbers are unified multiplicities that we use to count with. Thus even the representation ‘6’ is iconic in this respect: it represents the unity of the number six. As we saw in earlier chapters, the graphic space around the representation of a molecule, as well as a schematic representation of its symmetries by a geometrical figure, indicate that we are investigating a unified whole.

An even more iconic representation of the concurrent unity and multiplicity of a natural number is the following, which is the kind of representation used by Leibniz in various manuscript discussions of the foundations of arithmetic. {7}

|_____|_____|_____|_____|_____|_____|

This representation exhibits the unity of the number more strongly and positively than the set of strokes, because of the irreducible unity of the continuum represented by the line. It insists upon the unity of a number. Insofar as a natural number is a multiplicity of units taken together as one intelligible thing, iconic representations of a natural number must represent the distinctness of the component units by spatial side-by-side-ness while its unity may be represented strongly by the continuum (whose own unity is so strong) or more weakly by the spatial isolation of the units taken together.

One thing *inter alia* that this example shows is that representation cannot be explained in terms of merely physical tokens, even in the very simplest cases. (Here I echo a point made by Charles Parsons in his essay ‘The Structuralist View of Mathematical Objects,’ though I of course do not espouse structuralism.) {8} The Leibnizian representation just given is, in its iconic intent, not a representation of the number 6 by a merely physical line (which is not strictly continuous on either the macroscopic or microscopic levels), but by the continuum, an intelligible thing whose unity it invokes and requires. And the spatial side-by-side-ness that represents multiplicity is not just physical but also intelligible spatiality, a condition of the difference of different things. Elsewhere, I have argued that this representation is a standing structural analogy designed to exhibit a condition of the intelligibility of a natural number. {9}

The two representations of 6 just given will not, however, properly represent the additive structure of the natural numbers. This is because each in its own way represents the natural numbers as an uninflected sequence, stretching out towards infinity:

|_____|_____|_____|_____|_____|_____|_____|____...

/, //, ///, ////, /////, //////, …

Because every natural number is distinguished from its predecessor in exactly the same way (by the spatial concatenation of |_____| or /), every fact of arithmetic (like 2 + 4 = 6) will have to be recorded separately: there will be no possibility of a finite arithmetical table. In the first case, the arithmetical 'table' would just be the line divided into units, stretching out to infinity, all over again. In the second case, it would be a stroke table that extends infinitely in both dimensions.

	/	//	///	////	/////	//////	.	.	.
/	//	///	////	/////	//////	///////			
//	///	////	/////	//////	///////				
///	////	/////	//////	///////					
////	/////	//////	///////						
/////	//////	///////							
.									
.									
.									

The problem is that both these tables are unsurveyable and uninformative. In his book, *Laws and Symmetry*, Bas van Fraassen defends his semantic approach to philosophy of science by arguing that scientific investigations are centrally concerned with models, and that considerations of syntax aren't meaningful apart from semantics. The construction of models, he argues at length, depends on the selection of significant features of a system and associated symmetries or periodicities, that is, a group of transformations that leave that feature invariant. {10} I have argued earlier in this book that such semantic considerations must be supplemented by pragmatic considerations arising from the problem-solving context of use; and that models *qua* representations shade into nomological machines *qua* interventions, and that paper tools play an important role in this middle ground. The situation here can be explained in these terms.

Arabic numerals, considered as a representation of the natural numbers, introduce periodicities based on 10, 10^2, 10^3, and so forth, which are compounded or superimposed by means of horizontal and vertical conventions of bookkeeping. Thus we add:

$$\begin{array}{rr} & 123 \\ + & \underline{237} \\ & 360 \end{array}$$

Imagine if along a wall, for example, we recorded 123 strokes, and then recorded another 237 strokes, and then looked at a 'big enough' table like that given above to see the result of adding 123 + 237. It would be highly determinate in one way (we could see all the strokes) and highly indeterminate in another way (what systematic sense can be made of it?). My point is that the notation of Arabic numerals has a conceptual function that is not merely syntactic, but semantic and pragmatic as well: it creates a model of the natural numbers that precipitates a nomological machine, furnishing a 10 x 10 table for arithmetic, which summarizes most humanly relevant arithmetic facts of addition in 100 entries. It also exhibits, in virtue of its compounded 10^n periodicity, patterns that form the basis for problems concerning natural numbers, and their solutions.

This is one reason why notation is so important in mathematics, because of its role in creating models and precipitating nomological machines. Philosophers of mathematics have not clearly recognized this role perhaps because they have been so fixed on symbolic representation and so inattentive to iconic representation, and the iconic (and indexical) aspects of symbolic representations. This has also led them to posit an artificial disjunction between syntax, semantics, and pragmatics. Iconic representations need not be pictures of things with shape, though of course they often are; they may also be representations that exhibit the orderliness that makes something what it is. Arabic numerals, by exhibiting a certain multiply-periodic structure in the natural numbers, construct a model of them that allows for the articulation and solution of problems relating to the operation of addition. Here the role of syntax, semantics, and pragmatics cannot be disentangled; arithmetic is a domain in which form has content. Note that this doesn't mean that other notations aren't possible for the natural numbers, or that they can be identified with any one 'best' notation, or that there is a set of all possible notations, or that the notations exhaust what the natural numbers are.

The 'ruler' representation does not allow for the representation of multiplication at all. The stroke representation will represent the factorization of a natural number if we introduce conventions of vertical as well as horizontal grouping, but that introduction itself precipitates periodicities in the internal structure of the number: Is // repeated? Is /// repeated? How often? This convention iconically represents something about the multiplicative structure of a number, since multiplication is the iteration of addition.

4	5	6		7	8		9
//		//	///		//	////	///
//		//	///		//	////	///
		//			//		///
					//		

But what would a multiplication table look like in the stroke notation? Like the arithmetic table, it would go on and on, with an infinite number of entries, determinate but uninformative.

/	//	///	////	.	.	.
/	//	///	////			
//	//	///	////			
	//	///	////			
///	//	///	////			
	//	///	////			
	//	///	////			
////	//	///	////			
	//	///	////			
	//	///	////			
	//	///	////			
.						
.						
.						

Note that the entries	//	and	///
	//		///
	//		

are distinct, and there is no structural connection made between them.

By contrast, Arabic numerals furnish a 10 x 10 table for multiplication (where 2 x 3 is immediately 3 x 2, because they are both 6), and enhance the investigation of that most important feature of the natural numbers, the possibility of expressing each one as a unique product of primes. This is, of course, the insight on which much of number theory turns. Moreover, a great deal of modern number theory also consists of discerning and then imposing further symmetries or periodicities that hold of the natural numbers as well as further number systems in which they may be embedded. The *notation* makes possible procedures that allow for the investigation of the natural numbers, in terms of difficult but answerable problems. Another way to put my point is this: the natural numbers, represented by e. g. the stroke notation, are a highly determinate but infinitary domain, and insofar as they are infinitary, they are intractable. No problems can be posed with respect to them. This doesn't mean that they aren't intelligible; they are potentially intelligible in many ways, but we have to render them intelligible in at least a few ways in order to pose and solve problems about them. Our mathematical notations help us to do this; through the discernment and articulation of symmetries, they constitute models that are finitary and nomological machines that help us to solve problems. By saying this, I hope to avoid both structuralism on the one hand, and logicism as a kind of dogmatic Platonism on the other. I want to avoid an account that says that the natural numbers are empty placeholders, as well as an account that says that everything that is true about the natural numbers is always already true, in some big theory in the sky. The natural numbers are so determinate that whatever comes to be true of them will be necessarily true of them, but there is no sum total of all the true things that can be said of them. Different notations reveal different aspects of things. A similar argument could be made concerning the continua of geometry, whose infinitary determinateness is made finitary and tractable by geometrical figures, and the diagrams, lettering and instructions of geometers that record them.

3. The Application of Logic to General Topology

These observations explain a peculiar feature of the fate of mathematical logic in the twentieth century. Very few mathematicians were interested in Russell's project of rewriting the objects of other areas of mathematics in the notation of mathematical logic. However, they had some interest in the results of mathematical logic as a new domain with its own peculiar objects, and in the study of various mathematical domains in terms of patterns of inference, because different kinds of items leave their traces on the patterns of inference about them. (So logic, at least pursued as model theory, isn't so pure after all.) Frege's and Russell's notation for predicate logic was developed in relation to concepts, where a concept is conceived extensionally as the set of discrete items falling under the concept; and in relation to propositions, where what might count as a proposition is formalized in terms of recursively defined well formed formulae. This notation, not surprisingly, thus helped to precipitate set theory on the one hand and the theory of recursive functions on the other. It also proved strikingly un-illuminating when applied to other mathematical domains that were already well supplied with their own models and modes of representation: number theory, geometry, topology, analysis, and so forth. In what follows, I will give examples that illustrate both sides of this particular coin: the initially rather fruitless application of logic (in this case, the propositional logic that underlies predicate logic) to general topology; and the use of recursion theory to classify the hierarchy of sets in topological contexts. In mathematics, syntax, semantics, and pragmatics—formal languages, the models that construct the determinate yet infinitary as something finite and thinkable, and the problem-solving strategies that develop languages and models—are so intertwined that the fate of mathematical logic is not surprising. The discipline of model theory plays a special role within mathematical logic, however, to which I will return at the end.

Early twentieth century mathematical logic, under the influence of Hilbert's foundational program, was preoccupied with first order theories and computability. Mathematicians approached problems in the field of logic through the methods of recursion theory and the study of finitistic, combinatorial methods. This approach was quite fruitful, as the results of Lowenheim, Skolem, and Gödel show. However, it is also true that between Gödel's results in the 1930's and Cohen's results in the 1960's, the development of mathematical logic was slow. While the restriction of methods of proof to elementary finitistic ones may have provided clarification of certain aspects of logic, in other respects it introduced artificial complications and limited the development of logic. {11} Because in mathematics the modes of representation that play the role of syntax also precipitate models and problem-solving strategies, it is not surprising that mathematical logic, despite its claims to universality, was in fact tied to a certain domain of mathematical items.

However, another way of viewing logic revitalized the field in the work of Tarski and Stone. The set of all formulae of a formalized theory in predicate logic can be viewed as an algebra, and in general an algebra with infinite operations. When the formulae have been 'modded out' (organized symmetrically or periodically) into appropriate equivalence classes, they become one of a variety of lattices, Boolean, pseudo-Boolean, quasi-Boolean, topological Boolean, and so forth. This alternative interpretation thus views the formulae of logical calculi as mappings into certain lattices, which is a generalization of the truth table method, moving from two truth values, to an arbitrary finite number of truth values, to an infinite number of them. Topology enters here, because the open sets of a topological space form a (usually infinite) lattice to which formulae can be mapped; thus a topological space might be considered as an interpretation, a lattice of truth values, for logic. The approach of Frege and Russell, so fastened on arithmetic, showed no obvious rapprochement between logic and topology, but this approach made it seem promising, with topology provided a kind of concrete semantics for various logics.

And indeed it did eventually prove fruitful, shedding new light on previous results in mathematical logic, though its usefulness for topology is not so clear. The completeness of predicate logic was proved to be equivalent to Stone's Theorem on the representation of Boolean algebras. Gödel's Completeness Theorem for predicate logic was proved equivalent to a modification of the Stone Representation Theorem, as well as a result of the Baire Theorem on sets of the first category in topological spaces. {12} Moreover, the interpretation of logic in terms of algebra, lattice theory and topology, because of the way in which it diverged from the arithmetization of logic, led to the posing and solution of different sorts of problems. They included the results of Scott and Solovay on Boolean-valued models, which played an important role in their re-writing of Cohen's proof of the independence of the Continuum Hypothesis. {13} Generally speaking, the interpretation of logic by topology was at first rather sterile, because the application of logic was inappropriate—it had not been designed for the study of topological spaces. However, eventually the attempt to bring logic into rational relation with topology precipitated new kinds of models and problems, and moreover precipitated new forms of logic, as well as giving new life or prominence to items, problems, and logical forms earlier regarded as peripheral.

During the nineteen twenties and thirties, Alfred Tarski used logic as a means for representing and investigating the fine structure of topological spaces in general topology. The crucial insight governing his work is that propositional logic (and its variants), Boolean algebras (and their variants) and topological spaces can all be construed as lattices. Propositional logic, of course, underlies predicate logic. The propositional calculus can be considered a Boolean algebra; the ordinary method of truth tables yields homomorphisms which map the well formed formulae to the Boolean algebra consisting of two elements, 0 and 1. A formula is a tautology if it is mapped to 1 by every such homomorphism. Appropriate homomorphisms can be found which map the propositional calculus in this way to any Boolean algebra. In other words, any Boolean algebra, even an infinite one, can serve as a system of truth values for the propositional calculus.

The open sets of a topological space form a lattice, in fact, a complete lattice. So too do the closed sets, and the collection of closed and open sets. Thus, a topological space whose lattice of open sets forms a Boolean algebra can serve as a system of truth values for the propositional calculus. The answer to the question 'for which topological spaces does the lattice of open sets form a Boolean algebra?' depends on the way in which the Boolean operations are correlated with topological operations.

In an early paper written in 1937, Tarski correlates $X \vee Y$ with the ordinary set theoretic sum of open sets, $X \wedge Y$ with the ordinary set theoretic product, and $\sim X$ with the complement of the closure of X, so that operations on open sets again produce open sets. {14} Unfortunately, in this instance only isolated topological spaces emerge as truth value systems for the propositional calculus. Tarski observes that this interpretation of the propositional calculus is a ‘quite trivial and in fact a general set-theoretical (not especially topological) interpretation…; every set S can in fact be made into an isolated topological space by putting X = X for every X   S.’ This construction is equivalent to giving a set the discrete topology, which is topologically uninformative. Disappointed with this result, Tarski suggests that one might adjust the terms of the correlation, placing restrictions on the collection of open sets.

W. H. Stone, a functional analyst with interests similar to Tarski’s, lays out a general theory in which Λ, V, and ~ correspond to ordinary set-theoretic multiplication, addition, and complementation; the clopen sets of any topological space under these operations form a Boolean algebra. {15} Thus Stone focuses attention on topological spaces with a basis of clopen sets. These spaces are of genuine interest to both the topologist and the logician; known as Boolean spaces, they are totally disconnected, compact Hausdorff spaces. The best-known example is the Cantor space, which can be constructed by giving the two-member set (0,1) the discrete topology, and then giving the Cartesian product of denumerably many copies of it the product topology. It can also be thought of as the collection $2^{\mathbf{N}}$ of all characteristic functions of the set **N**, the natural numbers; this is a metric space whose metric induces a topology equivalent to the product topology. The Cantor set had been identified long before Stone’s researches as the following construction. Begin with the unit interval and remove the interval (1/3, 2/3), that is, the middle third; then remove the middle third (1/9, 2/9) and (7/9, 8/9) from the two remaining intervals; then the middle third from the four remaining intervals; and so forth. When this procedure is carried out denumerably many times, the Cantor set is the set of points that remains. (A topological space is a Cantor space if it is homeomorphic to the Cantor set.) Up until the debut of Stone’s researches, it had been regarded as an isolated curiosity; Stone shifted it to a position of much greater interest. One reason why the Cantor space is important is that it is both compact, and the product of discrete spaces; in mathematics, compactness and discreteness rarely arrive in the same package. The Cantor space is moreover dual to Euclidean space.

The main result in Stone's first article is the Stone Representation Theorem. It asserts a correspondence, and in fact a duality, between any Boolean algebra and some Boolean space. This theorem would not have arisen from either Boolean algebra or topology alone, but refers to and extends both domains. The correspondence posited by the Stone Representation Theorem between Boolean algebras and topological spaces is the deepest so far, for it selects out a class of topological spaces, Boolean spaces, which have an intrinsic interest for topology, analysis, and logic. Moreover, from it a wealth of special correspondences follow, so that knowledge of the fine structure of a Boolean space yields information about the fine structure of a Boolean algebra or ring, and vice versa. For example, a Boolean algebra is atomic if and only if the isolated points are dense in its dual space; finite if and only if it is the dual of a discrete space; countable if and only if the dual space is metrizable. This duality between spaces and rings has been generalized in a variety of ways and plays an important role in contemporary algebraic geometry.

The usefulness of this correlation is also apparent when it is applied to complete Boolean algebras, where every subset has both a supremum and an infimum. The duals of such algebras are called complete Boolean spaces; from the topological point of view, they seem at first pathological, since the closure of every open set is open, and the interior of every closed set is closed. It might be questioned whether any such spaces exist. However, the Stone Representation Theorem shows that there are 'many' of them, since each complete Boolean algebra corresponds to one of them, and every Boolean algebra can be completed. What might have been constructed with difficulty and only in isolated cases in point set topology has here been generated systematically, abundantly, and simply. As noted above, these spaces, and indeed Stone's work in general, play a central role in Scott and Solovay's version of Cohen's proof of the independence of the Continuum Hypothesis from the other axioms of set theory. Stone's work also underlies other important results: alternate proofs of various fundamental metatheorems concerning predicate, modal, and intuitionistic logics; Łos's concept of an ultraproduct of models; cylindric and polyadic algebras; and others. Note that these results are important for mathematical logic rather than for topology. {16}

Significantly, Stone was dissatisfied with his results, for he had hoped to encompass many more kinds of topological spaces, and find out more about their fine structure by his methods. In his 1937 paper, he wrote, 'Plainly we are engaged here in building a general abstract theory, and must accordingly be occupied to a considerable extent with the elaboration of technical devices. Such preoccupation appears the more unavoidable, because the known instances of our theory are so special, and so isolated that they throw little light upon the domain which we wished to investigate.' {17} Stone's work is interesting to us here not only for its results but also for its limitations. In the end, only a restricted portion of general topology was amenable to direct correlation with propositional logic. Stone had hoped to find more global representations, encompassing many more kinds of topological spaces and yielding significant information about their fine structure; in this he was disappointed. The research program that followed upon Stone's founding papers of 1936 and 1937 attempted the interpretation of logic by topology by using stronger and non-classical logics: predicate logic, modal logics, logics with set and function variables, logics with generalized quantifiers, and intuitionistic logic.

4. Logical Hierarchies and the Borel Hierarchy

The second case I want to discuss is the succession of attempts to find a precise analogy between hierarchies of formulae of predicate logic, representing recursive sets and their complements and projections, and the Borel hierarchy, a topological structure developed by Borel, Baire, and Lebesgue at the end of the 19th century. Borel sets are a family R of subsets of any given topological space, which includes the open sets and countable unions and intersections built up from them. Recursion theory serves as the link between the logical and the topological elements.

Mostowski, Kleene and Addison proposed ever more refined versions of this correlation, each a strengthening and correction of those preceding.

The recursive hierarchy begins with the ‘arithmetical hierarchy,’ which begins with recursive relations of *n*-tuples of integers, represented by formulae of predicate logic consisting of predicate letters. Recursive relations are those for which one can compute in a finite number of steps whether or not an *n*-tuple of integers belongs to it. This level is nicely represented by first order formulae consisting of predicate letters, since there are denumerably many recursive relations, so that denumerably many predicate letters can name them all. The next level, represented by predicate letters preceded by a finite number of existential quantifiers, consists of the recursively enumerable (*r.e.*) relations, projections along the *n*th coordinates of recursive relations. The hierarchy continues via successive applications of projection and complementation to relations at lower levels, represented by adding blocks of existential and universal quantifiers before the predicate letters (since = ~ ~ and ‘~’ represents complementation). Thus increasing complexity in the formulae mirrors increasing complexity in the sets of integers. In this case, one employs a two-sorted logic with set or function variables as well as individual variables, only the latter being quantified over. The ‘analytical hierarchy,’ which extends the arithmetical, allows projection along function coordinates as well, and is represented by second-order logic.

The Borel hierarchy is a family *R* of subsets of any given topological space, beginning with the open sets and continuing with the construction of countable unions and intersections. Ordinal numbers α classify Borel sets into levels R_α. The family of all open sets is level R_0; for $\alpha = \lambda + n > 0$, λ a limit ordinal, R_α is the family of all sets of the form X_k or X_k, according as *n* is even or odd, and the sets X_k belong to earlier levels. The Borel hierarchy continues upwards into the hierarchy of projective sets, whose first stage consists of Borel sets of finite and transfinite levels, and whose second stage consists of ‘analytic sets’.

In his paper “A Symmetric Form of Gödel’s Theorem,” Kleene remarks that Mostowski in 1946 had proposed an analogy between recursive sets and Borel sets, *r.e.* sets and analytic sets, and in general between the arithmetical hierarchy of recursion theory, and the Borel and projective hierarchy of topology. {18} The analogy is apparently thoroughgoing, since adjacent levels of both the arithmetical hierarchy and the topological hierarchy are related by a projection. For example, if a set is *r.e.*, then there is a recursive set such that the first is a projection of the second. Similarly, the sets of the projective hierarchy are obtained by the operation of ‘generalized projection’ from the Borel sets. However, Kleene asserts that the analogy is imperfect. A theorem of Lusin states that two disjoint analytic sets can always be separated by a Borel set, but two disjoint *r.e.* sets may not always be separated by a recursive set. The correlation therefore had to be completely revised, so that the arithmetical hierarchy of recursion theory corresponded only to the finite Borel hierarchy, and the analytical hierarchy of recursion theory corresponded to the projective hierarchy of descriptive set theory.

Kleene’s analogy is also unsatisfactory, however, for he was primarily interested in classifying definable sets of natural numbers. The difficulty is that there is a kind of incompatibility between the objects of recursion theory, the integers, and the canonical objects of topology, sets of real or complex numbers. (The Borel hierarchy is usually constructed upon a complete metric space like the intervals of reals between 0 and 1.) Since these items are so different—discrete, incomplete, and ‘first order’ in the former case but connected, complete, and ‘second order’ in the latter—some kind of mediating thing is required that admits an interesting topology, retains certain features of the reals, and exhibits recursive structure. Some adjustments in the recursive and topological hierarchies are also necessary. The following versions of the analogy were developed along these lines by John W. Addison in his doctoral dissertation and a paper that appeared in 1959. {19} [Figure 10.1]

The first of these compromise items is, not surprisingly, the Cantor space. The topological side of the analogy is the effective finite Borel hierarchy, constructed upon the Cantor space, where unions and intersections are limited to being recursive. Here the Borel hierarchy is altered to accord better with its recursive analogue. The first level of this hierarchy can then be thought of as the clopen sets that form a basis for the Cantor space, and the second level as the open sets. The recursive side is the arithmetical recursive hierarchy of sets *of sets* of integers, which requires set variables as well as individual variables for its expression. This analogy is successful to the extent that any set of sets of integers which occurs at the *n*th level of the recursive hierarchy will also occur at the *n*th level of the topological hierarchy; but it is limited, because sets which occur at the *n*th level of the topological hierarchy may not occur at the *n*th level of the recursive hierarchy.

A second mediating item is the Baire space, ω^{ω}, formed by giving the product of *N* copies of *N* the product topology. Although complete and with a basis of clopen sets, it is not compact; it can be identified with the irrational real numbers between but not including 0 and 1. The points in the product space can be thought of as functions from *N* to *N*, and the basis of clopen sets as composed of all such functions whose values at a finite number of integers are specified. The clopen sets are recursive in the sense that they can be specified by a finite amount of information. In this case, the topological side of the analogy is the Borel hierarchy constructed on the Baire space, called the finite hierarchy of Borel sets on the irrationals. The recursive side is the arithmetical hierarchy of sets of functions of integers, which requires function variables as well as individual variables for its expression. This analogy is limited by the fact that classes that occur at the first level of the recursive hierarchy occur at all levels of the Borel hierarchy. However, though it doesn't capture properly many second order topological properties, this analogy does yield effective versions of a number of theorems in consequence of the final and most successful version of the analogy, set forth in the table in [Figure 10.2].

The recursive side of this last analogy consists of the arithmetical hierarchy of sets of *functions of* integers, not just recursive, but recursive in a real. (A function recursive in a set S can be computed by a Türing machine with an oracle, a black box that can tell what integers are in S.)

The topological side consists again of the Borel hierarchy on the Baire space. This time the analogy is extremely thoroughgoing; it gives classes for all levels that occur at the same level in both the recursive and topological hierarchies. It can be extended to include the analytical hierarchy as well, so that the analogy in its fullest form looks as it is represented in the table given in Figure 10.2. Addison's last analogy provides the finest correlation of structure between the two hierarchies and thus leads to deep results like the common validity of Lusin's Separation Theorem and the Interpolation Theorem for a generalized version of predicate logic ($L_{\omega\ \omega}$) for both the recursive and the topological structures. More generally, over the years these analogies developed into a comprehensive theory which yields in a unified manner both the classical topological results and the theorems of the recursion theorists, and which serves as an important stage in the broader development of descriptive set theory.

Addison's work modified both sides of the original analogy by assimilating one to the other. This process of assimilation produces and provides a context for hybrids. Addison makes use of the Baire space, a topological space that mimics the reals but has recursive structure; other topological items modified to have recursive structure, like the effective Borel hierarchy; and recursive structures modified to approximate the complexity of the continuum, like the hierarchy of sets of functions recursive in a real. By means of his correlation, Addison produces models to which neither predicate logic nor topology alone would have given rise. Given that the integers are the canonical items of recursion theory, and the reals those of topology, Addison's analogy would have foundered without hybrid items to work on. The integers can be given no suitable topology in this context, for if the clopen sets are to be the recursive sets, then the toplogy will be just the discrete topology, since all finite sets are recursive. But then not just all countable unions of recursive sets, but all sets, will be open and correspondingly *r.e.*, and the analogy fails. The reals with their natural topology of open balls are also recalcitrant: since they are connected, they won't have any clopen sets beside the empty set and the whole space.

The work of Mostowski and Kleene in this episode shows that the iconic dimension of symbolic notation need not be figural. Some mathematical things cannot be used very well to 'fit out' (to use Nancy Cartwright's phrase) certain abstract structures expressed in a certain notation because, as in the case of topological spaces here, they are not well described by it. This means that the notation is designed for another kind of thing, that is, the notation, in this case logical notation, is not as subject matter-free as it pretends to be. First order predicate logic is 'designed for' sets of natural numbers, or, more generally, sets of integers. The attempt to bring logic into rational relation with topology requires the mediation of novel hybrids. It also leaves its mark on logic, as the last section will show.

A further iconic aspect of the logical symbolism employed here is evident in my summary table [Figure 10.2]. The horizontal correspondences exhibit the correspondences between models (the Borel hierarchy erected on the Baire space, the arithmetical and analytical hierarchy of sets of functions recursive in a real) iconically. The vertical articulation is also meaningful, and allows for the rational relation of the finite and the infinite, as well as increasing degrees of infinitary complexity. It exhibits the mathematicians' well-founded belief that we can move from the finite to the transfinite and moreover articulate the transfinite, iconically, giving a 'place' to mathematical entities and processes that only 150 years ago were off the page. In doing so, the table asserts a structural analogy among the finite, the transfinite, and the differentiated levels of the transfinite. The relation of the first to the second level of the arithmetical hierarchy is like the relation of the first to the second level of the analytical hierarchy, for example, and the similitude holds on both the logical and the topological sides.

5. Model Theory and Topological Logics

The original goal of Tarski and Stone was to find a representation by means of which logic could express the features of many different kinds of topological spaces, not just one special structure (like, e.g., the Baire space). I now turn to a different strategy for representing topological spaces, a model-theoretic approach that uses what are called topological logics. They are various modifications of predicate logic that inhabit a middle ground between first and second order predicate logic; one of them even has a generalized quantifier Q which, as the prefix of a formula, indicates that the set of individuals satisfying that formula is, in the cases given below, open.

Model theory is the study of the relation between a formal theory (most often, a first order theory) and its models, in the hope of exhibiting something useful to mathematicians who work in the target field indicated by the models. At this point in time, topological model theory has mostly been forgotten, perhaps because it never did find important applications in toplogy. (More specific developments have brought about some rapprochement between logic and analytic geometry. {20}) Model theorists tend to restrict their attention to first order theories because they are well-behaved enough to be governed by a series of important theorems, and, ironically, because they admit a wealth of non-standard models. (Abraham Robinson's book *Non-Standard Analysis* devotes a chapter to general topology, and another to topological groups. {21}) Precisely because first order theories are not categorical, they give the model theorist material to work with in the comparative study of the models of a theory. First-order theories of topology cannot capture the centrally important items and features of topology categorically, that is, up to isomorphism, but they do fall under the aegis of the important meta-theorems that define logical 'good behavior,' that is, the formal properties of logical systems that make them tractable to study. These include Completeness, Compactness, Lowenheim-Skolem, Interpolation, and Ultraproducts.

The Completeness Theorem states that any consistent set of formulae is satisfiable, has a model. Equivalently, if a formula is satisfied in every model of a theory, then it is a logical consequence of the axioms of that theory. The Compactness Theorem is an immediate consequence of the Completeness Theorem, because deductive proofs are finite: it states that a set of sentences has a model if and only if every finite subset has a model. However, precisely because first order theories are compact, they cannot represent the topological notion of compactness. The compactness of first order logic implies the existence of an ultrapower (which I define just below), and we can use an ultrapower construction to build a non-compact extension of any model which is elementarily equivalent to it, that is, which satisfies exactly the same sentences. Thus, no set of sentences composing a first order theory can capture the topological notion of compactness. The Lowenheim Skolem Theorem shows that a first order theory cannot capture the cardinality of models. The Craig Interpolation Theorem says that when X is a logical consequence of Y, a third sentence Z can be interpolated so that X is a logical consequence of Z and Z is a logical consequence of Y. These theorems are used in the construction of model-completions.

The Ultraproducts Theorem involves the construction of new models by forming certain product spaces of collections of models. A filter D over a set I is a subset of the power set of I that includes I itself, all intersections of the elements of D, and all Z when, if X   D, X   Z   I. An ultrafilter is a maximal proper filter. Consider the Cartesian product of sets A_i indexed by I; it is the collection of all functions *f* mapping each i ε I to an element of A_i. Using the ultrafilter of I as an indexing set when all the A_i are the same, we can construct the Cartesian product of all the A, and this is then called the ultrapower of A modulo D. The Ultraproducts Theorem shows that when we construct the ultrapower of a model, we obtain a new model which is elementarily equivalent to the original, and in which the original can be embedded in a simple way. {22}

The limits of the expressive powers of first order theories with respect to topology are forbidding. Thus, model theorists may address the task of representing topology by using logics that are called weak models, a pairing of a first order model with a collection *q* of subsets of the domain A of the model. When *q* is a topology, the pair is a called a topological model. The notation in which these theories is expressed includes a generalized quantifier Q, satisfying certain conditions so that when *q* is a topology, the set of individuals represented by a formula preceded by the quantifier Q is an open set. Although there is no direct relation of inclusion between first and second order logic, we can say that weak logics are intermediate, because weak logics are more expressive and yet not all the good behavior of first order logic has been sacrificed.

Research into these languages and their models mostly exhibits the inevitable trade-off between expressiveness and good behavior. L(Q) is governed by Completeness, Lowenheim-Skolem, Compactness and an Ultraproducts Theorem, but does not have Interpolation or Beth definability. It can express T_1 separation and discreteness, but not T_0, Hausdorff, T_3 or T_4, or notions like 'compactness,' 'the interior of a definable set is not empty,' and 'open in the product topology.' L (Q^n) has Completeness, Lowenheim-Skolem, Compactness, and an *r.e.* set of valid formulae, though again Interpolation and Beth definability fail. Its expressive power is greater: Hausdorff is definable, though not T_3 or T_4, nor the notion 'compact.' {23}

A pair of related but stronger languages are L(I) and L(I^n), languages formed by adding an 'interior operator' I to predicate logic, where (Ix) φ defines the interior of the set of points defined by φ. This logic has Completeness, Lowenheim-Skolem, Compactness, and an *r.e.* set of valid formulae; because it is monotonic, it has Interpolation and Beth definability. It captures the notion of Hausdorff, but it does not capture T_3 and T_4 separation. It can be extended to L(I^n), in a way analogous to the extension of L(Q) to L(Q^n). This logic has Completeness, Lowenheim-Skolem, Compactness, and an *r.e.* set of valid formulae, Interpolation, and Beth definability, and can express Hausdorff separation. {24} A final logic worth mentioning from this series is L_{top}, based on a two-sorted first-order language with both individual and set variables but quantification only over the former. It has in addition a topology on the domain of the individuals, and a membership relation ε. This language is expressive enough to define even T_3 separation, but T_4 still escapes its scope, as does that of topological compactness. It has Completeness, Lowenheim-Skolem, Compactness, and an *r.e.* set of valid formulae, Interpolation, and an Isomorphic Ultrapowers Theorem. {25} None of these logics can express topological compactness, and it was suggested that a sheaf-theoretic, category-theoretic representation might be more successful.

A more recent project, the development of 'geometric logic' by Steven Vickers, has a close relationship to topology: the class of models for a propositional geometric theory is automatically a topological space. The definition of geometric logics is, however, rather 'weird,' to use its creator's word. It makes a hard distinction between formulae and axioms. A logical formula is restricted in the connectives it can use to conjunction, disjunction, equality and existential quantification; and an axiom for a geometric theory expresses relationships between formulae in the form 'for all *x*, *y*, and *z*, … (formula 1 – formula 2)'. The missing connectives can be introduced, without nesting, in axioms. Geometric logic also allows infinite disjunctions in formulae. However, the main applications of this logic lie in theoretical computer science. Topology proves ultimately quite resistant to representation by predicate logic. {26} None of these languages seems to have proved fruitful in the work of topologists. The philosophical interest of this case study lies partly in this negative fact, and partly in the constitutive ambiguity of the various logics that have been investigated. On the one hand, the formal language considered symbolically is 'about' topology; on the other hand, the formal language considered iconically represents itself as a mathematical system, which does or doesn't exemplify the great meta-theorems of predicate logic.

6. Coda

Logic is one of the pillars of philosophy, but it is not the only one. I have often defended its rights in my own department. However, if we think about the project of logic in light of the case studies in this book, a fundamental conflict within that project emerges, summed up by the examples given in this chapter. Logic is the study of the rules of thought; it thus wishes to represent how we think independent of what we happen to be thinking about. Its modes of representation strive to be completely general and thus symbolic, so that it will not mistakenly offer the peculiar features of some subject matter or another as universal; its modes of representation must flee the iconic. The laws of logic should hold for thinking about anything, no matter what it is or where it is or when it shows up in history. Logic must then also formalize what we call subject and predicate, and then formalize the link between subject and predicate, and further between premises and conclusion; this means that it assumes a kind of abstracted homogeneity in the objects of its study reflected in its single, univocal idiom. These features of logic stem from its enterprise, to catalogue the laws of thought.

As any philosophical logician knows, these features of logic make it difficult for logic to accommodate reasonings that involve modalities (contingency, necessity, probability), or that are affected by the passage of time, or that concern individuals, or that require iconic displays, or that link heterogeneous acts of thought or speech or kinds of things. Either logic must be altered, and bear some of the marks of the peculiar objects or processes reasoning strives to cover, splitting into a family of heterogeneous idioms; or it must announce its limits. The point of this chapter, and my whole book, is to show that mathematicians typically reason about individuals as well as abstractions, refer successfully to specific things, link heterogeneous items, and exploit and construct ambiguity, in problem contexts that cannot escape geometry, modalities and historicity. Because they do this successfully all the time, it is no wonder that logic all by itself cannot express mathematics. When Paul Benacerraf limits the articulation of mathematical truth to logic and then complains that the ability of mathematicians to refer has been lost, it is no wonder; it is also no wonder that number theorists and geometers have not borrowed the language of logic to do their work. Mathematicians must and do employ a variety of modes of representation in tandem and in superposition, using natural language to explain what cannot be formalized: the relation of the idioms to each other, to the reader and to the regulative object. Moreover, as logic enters mathematics and becomes more and more mathematical, its iconic aspects (which in fact it has always had even when denying them) and useful ambiguities become more important, and it splits up into a family of no longer homogeneous idioms. This is one reason why the Aristotelian theory of the syllogism, and Stoic or propositional logic, being more purely logical and thus also more constrained, must persist alongside modern mathematical logic.

Chapter 10 Notes

1 *Principia Mathematica* (Cambridge: Cambridge University Press, 1910 / 1963), vol. 1, Part II, sec. A.

2 C. Chihara, *Ontology and the Vicious-Circle Principle* (Ithaca: Cornell University Press, 1973); an even more emphatic structuralism is presented in *A Structural Account of Mathematics* (New York: Oxford University Press, 2004).

3 P. Maddy, *Naturalism in Mathematics* (Oxford: Clarendon Press, 1997), 23.

4 Ibid., 33.

5 Ibid., 34.

6 Ibid., 26.

7 E. Grosholz, 'L'analogie dans la pensée mathématique de Leibniz' in D. Berlioz and F. Nef (eds.), *L'actualité de Leibniz: Les deux Labyrinthes* (Stuttgart: Steiner, 1999), *Studia Leibnitiana* Supplementa 34, 511-22.

8 C. Parsons, 'The Structuralist View of Mathematical Objects,' in *The Philosophy of Mathematics*, W. D. Hart, ed. (Oxford: Oxford University Press, 1996) 272-309. I have also learned a great deal from Michael Resnik's *Mathematics as a Science of Patterns* (Oxford: Oxford University Press, 1997). Indeed, I've developed my own position over the last twenty years in part by arguing against Donald Gillies' rather naturalist empiricism and Michael Resnik's structuralism, and in the meantime have come to value their friendship. The footnotes in this book don't really do justice to my intellectual debt to both these philosophers.

9 Grosholz, 'L'analogie dans la pensée mathématique de Leibniz'.

10 Van Fraassen, *Laws and Symmetry*, Ch. 10 and 11.

11 H. Rasiowa and R. Sikorski complain about this finitist tendency in logic in the preface to *The Mathematics of Metamathematics* (Warsaw: Polish Scientific Publishers, 1963).

12 Ibid., 1-6.

13 Dana Scott, ‘A proof of the independence of the continuum hypothesis,’ *Mathematical Systems Theory* 1 (1967), 89-111.

14 ‘Sentential Calculus and Topology,’ reprinted in *Logic, Semantics and Metamathematics: Papers from 1923 to 1938* (Oxford: Clarendon Press, 1956 / Indianapolis: Hackett, 1983, 421-454.

15 ‘The Theory of Representation for Boolean Algebras,’ *Transactions of the American Mathematical Society*, 40 (1936), 37-111; and ‘Applications of the Theory of Boolean Rings to General Topology,’ *Transactions of the American Mathematical Society*, 41 (1937), 375-481; see further discussion of this work in P. R. Halmos, *Boolean Algebras*, Mimeographed lecture notes, 1959 (held in the University of Chicago library).

16 See J. L. Bell, *Boolean-valued Models and Independence Proofs in Set Theory* (Oxford: Clarendon Press, 1977); and H. Rasiowa, *An Algebraic Approach to Non-classical Logics*, Amsterdam: North Holland, 1974.

17 Stone, ‘Applications of the Theory of Boolean Rings to General Topology.’

18 S. C. Kleene, ‘A Symmetric Form of Gödel’s Theorem,’ *Indagationes Mathematicae* 12 (1950), 244-266; A. Mostowski, ‘On definable sets of positive integers,’ *Fundamenta Mathematicae* 34 (1946), 81-112.

19 J. W. Addison, *On Some Points in the Theory of Recursive Functions*, Ph. D. Thesis Madison: University of Wisconsin, 1954; and ‘Separation Principles in the Hierarchies of Classical and Effective Descriptive Set Theory,’ *Fundamenta Mathematicae* 46 (1959), 123-135.

20 See L. P. D. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge: Cambridge University Press, 1998. A consequence of Abraham Robinson’s early work on model theory was the definition of o-minimality, a notion arising in logic that seems to correspond well to Grothendieck’s notion of a ‘tame’or well-behaved topology, where the investigation of canonical geometrical items takes priority.

21 *Non-standard Analysis* (Amsterdam: North Holland, 1966); repr. (Princeton: Princeton University Press, 1996).

22 For more detail, see C. C. Chang and H. J.Keisler, *Model Theory* (Amsterdam: North Holland, 1973 / 1977; Elsevier, 1990).

23 J. Sgro, ‘Completeness Theorems for Continuous Functions and Product Topologies,’ *Israel Journal of Mathematics* 25 (1976), 249-72; see also ‘Completeness Theorems for Topological Models,’ *Annals of Mathematical Logic* 11 (1977), 173-193.

24 C. C. Chang, *Modal Model Theory* (Berlin: Springer-Verlag Lecture Notes #337, 1973).

25 S. Garavaglia, ‘Model Theory of Topological Structures,’ *Annals of Mathematical Logic* 14 (1978), 13-37.

26 See, for example, *Topology via Logic*, Cambridge Tracts in Theoretical Computer Science 5 (Cambridge: Cambridge University Press, 1988).